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Modeling swelling instabilities in surface confined hydrogels

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Modeling Swelling Instabilities in Surface Confined Hydrogels

by

Abiola Shitta

A thesis submitted in partial fulfillment
of the requirements for the degree of
Master of Science in Chemical Engineering
Department of Chemical and Biomedical Engineering
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Dedication

I would like to dedicate this work to my father and mother to whom I am and always will be eternally grateful. They took the time to make sure that I had all the tools necessary to become the best that I could be. Had it not been for this, my life would be very different and I probably never would have lived up to my greatest potential.

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Modeling Swelling Instabilities in Surface Confined Hydrogels

Abiola Shitta

Abstract

The buckling of a material subject to stress is a very common phenomenon observed in mechanics. However, the observed buckling of a surface confined hydrogel due to swelling is a unique manifestation of the buckling problem. The reason for buckling is the same in all cases; there is a certain magnitude of force that once exceeded, causes the material to deform itself into a buckling mode. Exactly what that buckling mode is as well as how much force is necessary to cause buckling depends on the material properties. Taking both a finite difference and analytical approach to the problem, it is desired to obtain relationships between the material properties and the predicted buckling modes. These relationships will make it possible for a hydrogel to be designed so that the predicted amount of buckling will occur.

Background

Responsive polymers and their applications have become a very active area of research. The applications for these polymers range from biomimetic actuators to biocatalysts and the list continues to grow [1]. Responsive surfaces are another application of these materials where the fact that the polymer is attached to a surface can give rise to unique behavior [2, 3].

A responsive polymer hydrogel confined to a surface can be made to swell with a variety of different stimuli. This swelling can, under certain circumstances, cause the hydrogel to buckle. If the gel is free to swell as it pleases or is not constrained, there is no observed buckling phenomenon. However, confining the hydrogel to a surface causes stress to build up within the gel as it swells. This build up in stress is caused by the opposition to its free swelling due to the surface confinement. This opposing force manifests itself as a compression of the gel. If the magnitude of the compression is sufficient, buckling of the hydrogel will occur. Fig. 1 shows an illustration of this system with a uniaxial compression. This is the framework that will be used to model buckling in the hydrogel.

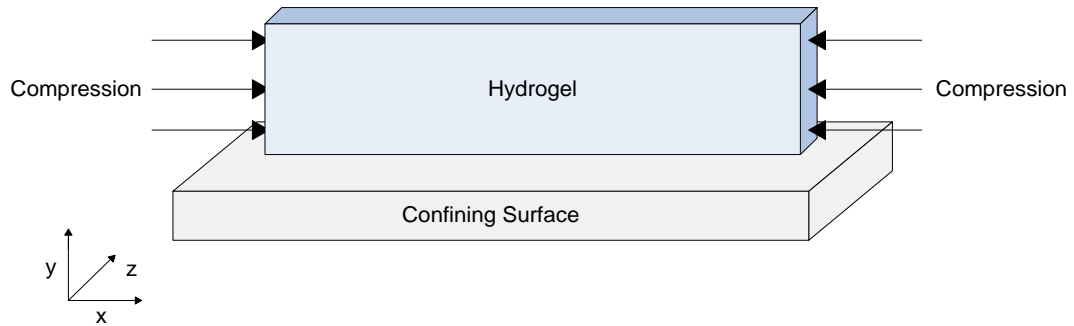


Figure 1: Setup of Buckling Problem

The stress at which buckling occurs depends on a variety of factors; the dimensions of the material, the way it is supported and the properties out of which it is made [4]. In the case of a swelling hydrogel, it could also be dependent on how much the hydrogel is allowed to or capable of swelling since this could very well determine the magnitude of compression that it undergoes.

It is the goal of this paper to find out how these factors play a role in the buckling of a hydrogel and potentially what relationships can be determined from these phenomena.

Continuum Mechanics

Continuum mechanics make it possible for a body being subject to stress to be analyzed with minimal complexity. A continuum body is described as a body in which the molecular interactions are ignored. If you were to continually break the body up into smaller and smaller pieces there will be a point at which the interactions between those pieces will be negligible. Those pieces will also eventually become approximate points which cover the entire volume of the continuum body. This idea makes it possible to solely analyze the effect of applying a load to a material without the added complexity of studying the intermolecular forces.

Within continuum mechanics there are principles that apply to all materials as well as the constitutive equations that define specific mechanical behavior. The general principles refer to things like the conservation of mass, energy and momentum. To our knowledge, all known materials obey these general principles; so the equations that they generate set up the foundation of continuum mechanics. The constitutive equations apply to materials that are linear elastic isotropic, linear elastic anisotropic, nonlinear elastic isotropic and so on and so forth. These descriptions tell exactly what modifications have to be made so that the general principles still apply and accommodate the behavior of the material.

The difficulty with problems in continuum mechanics, as with problems in any other field, lies primarily in how much information you have about the problem at hand. For example, let us say that you know the initial and final conformation of a material and you want to find the equations that describe what stresses are required to make that so. This problem would require that you have an equation that describes the displacement of the points in the body from the initial to the final state. This equation is difficult to come by but it could probably be determined by marking the material initially and observing how the same points move after loading to arrive at an approximate equation. The point is that the equations in continuum mechanics can be solved yet the necessary pieces may not always make it possible to achieve a solution.

At any rate, this analysis alone is not enough to describe the buckling behavior that is observed. While a buckled solution can result from a certain stress field being applied to a material, this tells us nothing about what causes it and the criteria necessary to make it so.

Linear Elastic Theory

A polymer hydrogel is an elastic material. It has the ability to return to its initial size and shape after it has been subjected to stress. This only applies up until the point before the material yields or buckles to the applied stress. Linear elastic theory is capable of describing the behavior of a material before it yields or undergoes a large deformation. If the material were to yield, its behavior then becomes nonlinear and a more complicated set of equations would be necessary to map the deformation of the material from that point on. Although this can be done, the approach typically is to initially use linear elastic theory. If this analysis fails to accurately describe the materials behavior then a nonlinear analysis is justified.

The equations used in linear elastic theory are linear relationships between the stress and strain of the material[5-7]. This along with the ability of the material to return to its initial state are the main tenets of the theory. The other tenets of the theory are that the rate at which the load is applied is of no consequence and the deformations that occur are very small provided that the material is still stable. With this added piece of knowledge, a framework can be devised that describes the behavior of a polymer hydrogel being subject to stress. The only issue is how to describe what happens to the material when it buckles without making use of the nonlinear analysis.

Structural or Elastic Stability Theory

Structural and Elastic Stability Theory refer to the same general idea. The idea is that there comes a point at which a material, be it elastic or not, loses its stability or its ability to support the applied load. This instability can make itself manifest in a variety of ways.

It can be done through bending as well as through multiple buckling modes. The point at which this instability occurs depends on how much work is being done on the material as well as the amount of energy required to cause a large deformation. This means that the nature of the material and the applied forces on that material determine how it looks when it becomes unstable and what critical values of these parameters are required to make it so.

This is a very crucial part in the study and analysis of buckling phenomena. This theory makes it possible to have tangible, applicable quantities that can be used to show how they affect buckling and structural stability.

Biharmonic Equations and the Airy Stress Function

The biharmonic equation is a differential equation that results from the equation shown below which uses the biharmonic operator ∇^4 .

$$\nabla^4 \psi = 0 \quad (1)$$

In Cartesian coordinates and in three dimensions the equation becomes:

$$\begin{aligned} \nabla^4 \psi &= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \psi \\ &= \frac{\partial^4 \psi}{\partial x^4} + \frac{\partial^4 \psi}{\partial y^4} + \frac{\partial^4 \psi}{\partial z^4} + 2 \frac{\partial^4 \psi}{\partial x^2 \partial y^2} + 2 \frac{\partial^4 \psi}{\partial x^2 \partial z^2} + 2 \frac{\partial^4 \psi}{\partial y^2 \partial z^2} = 0 \quad (2) \end{aligned}$$

There are certain complex phenomena that require the use of this equation[8]. The bending and flexing of thin flat plates is one of these phenomena. If a confined hydrogel strip is treated like a thin flat plate, then a form of the biharmonic equation can be used to analyze the stresses acting on it. This form of the biharmonic equation is called the Airy Stress function. This function is capable of determining the equilibrium conformation of a material subjected to stress. The main idea behind this function is that at

equilibrium, the stresses acting on the object satisfy a set of compatibility criteria[5, 7].

These criteria are shown below.

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z} + F_x = 0$$

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{zy}}{\partial z} + F_y = 0$$

$$\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + F_z = 0 \quad (3)$$

With these equations in hand, the Airy Stress function can be derived. This then makes it possible for the relationships that exist within linear elastic theory to be used to further simplify the equations so that there are fewer degrees of freedom.

Derivation of Airy Stress Function

The Airy stress function (ϕ) is a function that describes the equilibrium of a continuum body. In the absence of body forces, in two dimensions with plane strain, the strain components at equilibrium satisfy Eq. 4.

$$\frac{\partial^2 e_{xx}}{\partial y^2} + \frac{\partial^2 e_{yy}}{\partial x^2} - 2 \frac{\partial^2 e_{xy}}{\partial x \partial y} = 0 \quad (4)$$

From linear elastic theory, the stress strain relations are:

$$e_{xx} = \frac{1 + \nu}{E} [(1 - \nu)\sigma_{xx} - \nu\sigma_{yy}]$$

$$e_{yy} = \frac{1 + \nu}{E} [(1 - \nu)\sigma_{yy} - \nu\sigma_{xx}]$$

$$e_{xy} = \frac{1 + \nu}{E} \sigma_{xy} \quad (5)$$

Substituting the stress strain relations into Eq. 4,

$$(1 - \nu) \frac{\partial^2 \sigma_{xx}}{\partial y^2} - \nu \frac{\partial^2 \sigma_{yy}}{\partial y^2} + (1 - \nu) \frac{\partial^2 \sigma_{yy}}{\partial x^2} - \nu \frac{\partial^2 \sigma_{xx}}{\partial x^2} - 2 \frac{\partial^2 \sigma_{xy}}{\partial x \partial y} = 0 \quad (6)$$

The potential functions defined by the stresses are shown below.

$$\sigma_{xx} = \frac{\partial^2 \phi}{\partial y^2} \quad \sigma_{yy} = \frac{\partial^2 \phi}{\partial x^2} \quad \sigma_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y} \quad (7)$$

Substituting the potential function into Eq. 6,

$$(1 - \nu) \frac{\partial^4 \phi}{\partial y^4} + (1 - \nu) \frac{\partial^4 \phi}{\partial x^4} + 2(1 - \nu) \frac{\partial^4 \phi}{\partial x^2 \partial y^2} = 0$$

$$\frac{\partial^4 \phi}{\partial y^4} + \frac{\partial^4 \phi}{\partial x^4} + 2 \frac{\partial^4 \phi}{\partial x^2 \partial y^2} = \nabla^4 \phi = 0 \quad (8)$$

There are other biharmonic equations that expressed in terms of strains, stresses and displacements that describe a body in equilibrium[7, 9]. It is our desire to obtain the buckled conformation of the hydrogel and this will require that the equation be expressed in terms of displacements. This same principle is applicable to the boundary conditions and they will also need to be expressed in terms of displacements so that that will be the only unknown.

Finite Difference Approximation

The finite difference approximation method was used to solve the differential equation that models the buckling of the hydrogel. This method was also used to attempt to see if changes in the parameters of the function would give rise to noticeable changes in the buckling behavior. It has been implemented as a way to solve boundary value problems that contain partial differential equations of various types and orders[10-13]. The finite difference approximation breaks down derivatives into small and gradual step changes over a given variable. For example, the first derivative of a variable y with respect to x is shown below:

$$\frac{dy}{dx} \cong \frac{\Delta y}{\Delta x} = \frac{y(x + \Delta x) - y(x)}{\Delta x}$$

As the step size becomes smaller, the approximation of the derivative becomes more accurate. However, more calculations have to be done to achieve this accuracy. The effect of this accuracy on solution stability will be discussed in further detail in a later section. Below are the some of the finite difference approximations necessary to solve the Airy Stress function.

$$\begin{aligned} \frac{\partial^4 Z}{\partial x^4} &= \frac{Z_{i+2,j} - 4Z_{i+1,j} + 6Z_{i,j} - 4Z_{i-1,j} + Z_{i-2,j}}{(\Delta x)^4} \\ \frac{\partial^4 Z}{\partial y^4} &= \frac{Z_{i,j+2} - 4Z_{i,j+1} + 6Z_{i,j} - 4Z_{i,j-1} + Z_{i,j-2}}{(\Delta y)^4} \\ \frac{\partial^4 Z}{\partial x^2 \partial y^2} &= \frac{Z_{i+1,j+1} + Z_{i+1,j-1} - 2Z_{i+1,j} - 2Z_{i-1,j} + 4Z_{i,j} - 2Z_{i,j+1} - 2Z_{i,j-1} + Z_{i-1,j+1} + Z_{i-1,j-1}}{(\Delta x)^2 (\Delta y)^2} \end{aligned}$$

Combining these derivative expressions make it possible to have an approximation of the Airy Stress function or any other biharmonic equation as well as the behavior of the function on the boundaries. While further derivation may be necessary to describe the boundary behavior, they can be accurately encompassed within the framework of finite differences.

Basic Idea and Setup

Once the finite difference equations have been derived, the area over which the equation applies must be defined. The points of analysis on the hydrogel are along the surface in the x and y directions. The goal is to use the overall equation to solve for the deflection of the surface in the z direction. Fig. 2 shows how the grid points are defined as well as where the boundaries lie.

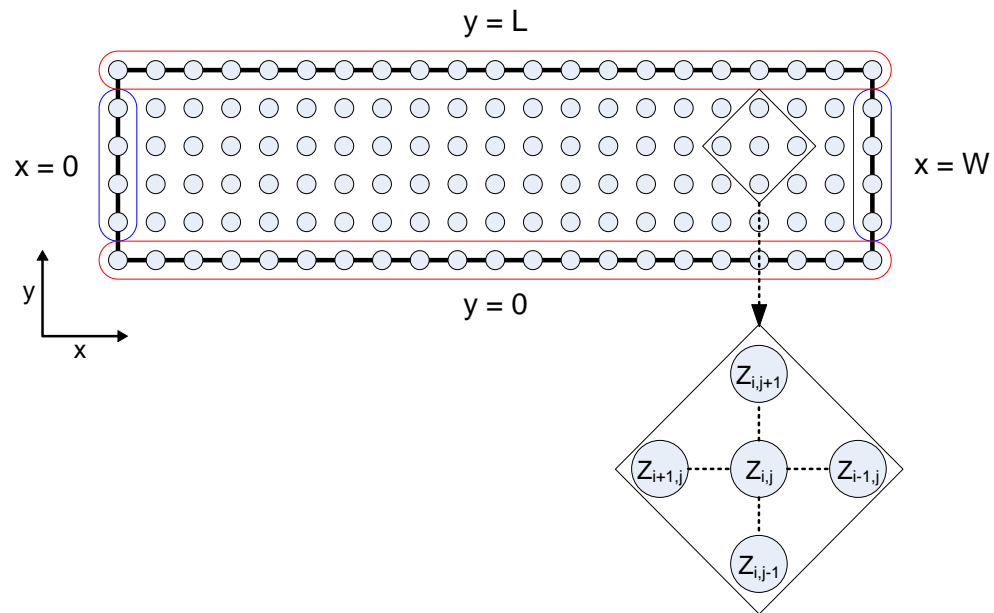


Figure 2: Boundaries and Grid Definition of Finite Difference Method

The points are counted from the bottom left when x and y are equal to zero. They are then iteratively stepped out in increments of the defined step size upwards in the x direction. Once the width of the hydrogel is reached, the next grid point is set at when x equals zero. However, that point is one step above the first grid point in the y direction. This procedure is continued until the points mark the boundaries and the entire surface at regularly spaced intervals.

Stability of Solution

The solution of the finite difference approximation has multiple limitations. Since this is an approximation, there will always be a certain amount of error. There is a point however at which this error becomes intolerable to the set of equations being approximated. At this point the system can go unstable and the solution that is calculated can be very far from accurate.

The von Neumann method is capable of determining the stability criteria for the finite difference equations over a specific area [10, 14]. This method introduces an error into the difference equations which is then subsequently solved for. The upper limit of this error is determined and this becomes the stability criteria. There seem to only be solutions to specific problems solved with this method and very little explanation about how to go about using the method itself. Because of this, the number of grid points, which determine the step size of that variable, were changed and the effect on stability was analyzed.

Boundary Conditions

Provided that the boundary conditions are linear in nature, a linear solving technique will provide a solution. However, non linear boundary conditions will require the use of a non linear solver. There are various ways to describe the boundaries of a material subjected to stress[6, 7, 9]. In terms of the vertical displacement in the z direction, the boundary conditions are defined as follows.

For a clamped edge,

$$Z = 0 \quad (9)$$

$$\frac{\partial Z}{\partial y} = 0 \quad (10)$$

Combining Eqs.9 & 10,

$$\frac{\partial Z}{\partial y} - Z = 0 \quad (11)$$

The forward finite difference approximation of the first derivative is:

$$\frac{\partial Z}{\partial y} = \frac{Z_{i,j+1} - Z_{i,j}}{(\Delta y)} \quad (12)$$

Substituting Eq. 12 into Eq.11, the expression for a clamped edge is:

$$\frac{Z_{i,j+1} - Z_{i,j}}{(\Delta y)} - Z_{i,j} = 0 \quad (13)$$

For a simply supported edge,

$$Z = 0 \quad (14)$$

$$\frac{\partial^2 Z}{\partial x^2} + \nu \frac{\partial^2 Z}{\partial y^2} = 0 \quad (15)$$

Combining Eqs. 14 & 15,

$$\frac{\partial^2 Z}{\partial x^2} + \nu \frac{\partial^2 Z}{\partial y^2} - Z = 0 \quad (16)$$

Expressed in forward finite differences with respect to x and in central finite differences with respect to y, Eq. 16 for a simply supported edge is:

$$\frac{Z_{i+2j} - 2Z_{i+1j} + Z_{ij}}{(\Delta x)^2} + \nu \frac{Z_{ij+1} - 2Z_{ij} + Z_{i,j-1}}{(\Delta y)^2} - Z_{ij} = 0 \quad (17)$$

For a free edge,

$$\frac{\partial^2 Z}{\partial y^2} + \nu \frac{\partial^2 Z}{\partial x^2} = 0 \quad (18)$$

$$\frac{\partial^3 Z}{\partial y^3} + (2 - \nu) \frac{\partial^3 Z}{\partial x^2 \partial y} = 0 \quad (19)$$

Combining Eqs. 18 & 19,

$$\frac{\partial^3 Z}{\partial y^3} + (2 - \nu) \frac{\partial^3 Z}{\partial x^2 \partial y} - \frac{\partial^2 Z}{\partial y^2} - \nu \frac{\partial^2 Z}{\partial x^2} = 0 \quad (20)$$

Expressing Eq. 20 with central finite differences with respect to x and backward finite differences with respect to y, the equation for a free edge is:

$$\begin{aligned} & \frac{Z_{ij} - 3Z_{i,j-1} + 3Z_{i,j-2} + Z_{i,j-3}}{(\Delta y)^3} \\ & + (2 - \nu) \frac{Z_{i+1j} - 2Z_{ij} + Z_{i-1j} - Z_{i+1,j-1} + 2Z_{i,j-1} - Z_{i-1,j-1}}{(\Delta x)^2 (\Delta y)} \\ & - \frac{Z_{ij} - 2Z_{i,j-1} + Z_{i,j-2}}{(\Delta y)^2} - \nu \frac{Z_{i+1j} - 2Z_{ij} + Z_{i-1j}}{(\Delta x)^2} = 0 \quad (21) \end{aligned}$$

The boundary conditions that were used as well as where they were used are highlighted in Fig. 3.

$$\begin{array}{c}
 y = L \quad \frac{\partial^3 Z}{\partial y^3} + (2-\nu) \frac{\partial^3 Z}{\partial x^2 \partial y} - \frac{\partial^2 Z}{\partial y^2} - \nu \frac{\partial^2 Z}{\partial x^2} = 0 \\
 \\
 \begin{array}{ccc}
 x = 0 & \boxed{\phantom{\text{Diagram}}} & x = W \\
 \frac{\partial^2 Z}{\partial x^2} + \nu \frac{\partial^2 Z}{\partial y^2} - Z = 0 & & \frac{\partial^2 Z}{\partial x^2} + \nu \frac{\partial^2 Z}{\partial y^2} - Z = 0 \\
 \\
 y & \begin{array}{c} y = 0 \quad \frac{\partial Z}{\partial y} - Z = 0 \\ \end{array} & \\
 \downarrow & \begin{array}{c} \\ \end{array} & \\
 x & &
 \end{array}
 \end{array}$$

Figure 3: Boundary Conditions at Specified Boundaries

At the lower y boundary, the hydrogel was treated as a clamped edge while at the upper y boundary, it was treated as a free edge. Both of the x boundaries were treated as simply supported edges. It was also necessary to introduce buckling on the upper y boundary in the form of a sine curve. Without this, no solution other than zero across the whole surface can be calculated. Zero is a very good solution to the biharmonic equation. Therefore a reason for any other solution must be justified and introduced through the relevant equations. While this induced buckling boundary condition may not accurately describe the actual buckling phenomenon, it sets up a condition which could mimic an approximation of this behavior.

Matrix Inversion as a Means of Solution

Once the set of equations have been generated to map the behavior of the hydrogel strip, a solution to the conformation of the buckled surface must be obtained. Since the set of generated equations are linear in nature, a linear solving technique will be able to arrive at a solution. Matrix inversion was the method of choice due to ease of programming as well as it being a built in function in MATLAB.

Matrix inversion simply requires there to be a coefficient matrix as well as a solution matrix for the defined system. The coefficient matrix comes from the equations that are derived and that govern the materials behavior. The solution matrix takes into account certain known values that may exist and integrates them into finding a solution. For example, if the coefficient matrix is defined as C and the solution matrix as S, the expression to solve for the variable X would be:

$$CX = S$$

Hence the solution with matrix inversion is

$$X = C^{-1}S$$

Once the solution was obtained, the values were then rearranged and plotted using the same grid point definition as highlighted in Fig. 2.

Analytical Solution

Using the analytical solution of the biharmonic equation is a viable way to achieve numerical solutions to the buckling problem. While the results obtained through this method may not be representative of what happens experimentally, it does provide an insight into the nature of the mechanics of a very simplified case of this phenomenon.

Separation of Variables

The separation of variables technique was used to solve for the analytical solution of the biharmonic equation. While other methods exist, I am most familiar with this method and the solution is much easier to construct. My initial attempt at using this technique was unsuccessful. After reading further, I discovered that the separation of variables procedure would have to be used multiple times in order to produce a solution. Otherwise, if one of the functions with respect to a certain variable is presumed, the separation is much easier. The general procedure for this function's variable separation is highlighted below.

Assuming Z is a function of x and y only:

$$Z(x, y) = X(x)Y(y) \quad (22)$$

Hence the biharmonic equation will yield

$$\frac{\partial^4 Z}{\partial x^4} + 2 \frac{\partial^4 Z}{\partial x^2 \partial y^2} + \frac{\partial^4 Z}{\partial y^4} = 0$$

$$\frac{d^4 X}{dx^4} Y + 2 \frac{d^2 X}{dx^2} \frac{d^2 Y}{dy^2} + X \frac{d^4 Y}{dy^4} = 0 \quad (23)$$

Dividing by XY:

$$\frac{1}{X} \frac{d^4 X}{dx^4} + 2 \left(\frac{1}{X} \frac{d^2 X}{dx^2} \right) \left(\frac{1}{Y} \frac{d^2 Y}{dy^2} \right) + \frac{1}{Y} \frac{d^4 Y}{dy^4} = 0 \quad (24)$$

New variables can be defined that can separate this function into different parts.

$$X_1(x) = \frac{1}{X} \frac{d^4 X}{dx^4}; \quad X_2(x) = \frac{1}{X} \frac{d^2 X}{dx^2}; \quad Y_1(y) = \frac{1}{Y} \frac{d^4 Y}{dy^4}; \quad Y_2(y) = \frac{1}{Y} \frac{d^2 Y}{dy^2} \quad (25)$$

The new expression that contains these variables is:

$$X_1 + 2X_2 Y_2 + Y_1 = 0 \quad (26)$$

Differentiating Eq. 26 with respect to x:

$$\frac{dX_1}{dx} + 2Y_2 \frac{dX_2}{dx} = 0 \quad (27)$$

Which simplifies to:

$$\frac{dX_1}{dx} / \frac{dX_2}{dx} = -2Y_2 = \text{constant} \quad (28)$$

Differentiating Eq. 26 with respect to y:

$$\frac{dY_1}{dy} + 2X_2 \frac{dY_2}{dy} = 0$$

Which simplifies to:

$$\frac{dY_1}{dy} / \frac{dY_2}{dy} = -2X_2 = \text{constant} \quad (29)$$

Using Eqn. 29 and defining a constant as λ^2 ,

$$X_2 = \frac{1}{X} \frac{d^2 X}{dx^2} = -\lambda^2 \quad (30)$$

Therefore:

$$\frac{d^2 X}{dx^2} + \lambda^2 X = 0 \quad (31)$$

Eq. 31 has solutions of the form:

$$X(x) = A \sin(\lambda x) + B \cos(\lambda x) \quad (32)$$

Using Eq. 32 and solving for Eq. 25 shows that,

$$X_1(x) = \frac{1}{X} \frac{d^4 X}{dx^4} = \lambda^4 \quad (33)$$

Substituting Eqs. 30 & 33 into Eq. 26 yields:

$$\frac{1}{Y} \frac{d^4 Y}{dy^4} - 2\lambda^2 \left(\frac{1}{Y} \frac{d^2 Y}{dy^2} \right) + \lambda^4 = 0 \quad (34)$$

Multiply by Y:

$$\frac{d^4 Y}{dy^4} - 2\lambda^2 \frac{d^2 Y}{dy^2} + \lambda^4 Y = 0 \quad (35)$$

The solution to Eq. 35 can be expressed in terms of exponentials, trigonometric functions or a combination of both. Different combinations have been used and in different ways [8, 9], but the y function in terms of exponentials can be expressed as:

$$Y(y) = C_1 e^{\lambda y} + C_2 e^{-\lambda y} + C_3 y e^{\lambda y} + C_4 y e^{-\lambda y} \quad (36)$$

Therefore, the overall expression for Z using Eqs. 32 & 36 is:

$$Z(x, y) = [A \sin(\lambda x) + B \cos(\lambda x)] [C_1 e^{\lambda y} + C_2 e^{-\lambda y} + C_3 y e^{\lambda y} + C_4 y e^{-\lambda y}] \quad (37)$$

From observed experiments it is fair to assume that the buckling modes take on a sinusoidal shape as a function of x. Hence the known function behavior will be a sine function with respect to x. Implementing this into the biharmonic equation in terms of displacement yields the following results.

$$\nabla^4 Z = \frac{\partial^4 Z}{\partial x^4} + \frac{\partial^4 Z}{\partial y^4} + 2 \frac{\partial^4 Z}{\partial x^2 \partial y^2} = 0$$

$$Z = f(x)f(y) ; f(x) = \sin(\alpha_m x)$$

$$\begin{aligned} \nabla^4 Z &= \alpha_m^4 \sin(\alpha_m x) f(y) + \sin(\alpha_m x) \frac{\partial^4 f(y)}{\partial y^4} - 2 \alpha_m^2 \sin(\alpha_m x) \frac{\partial^2 f(y)}{\partial y^2} \\ &= \sin(\alpha_m x) \left[\frac{\partial^4 f(y)}{\partial y^4} - 2 \alpha_m^2 \frac{\partial^2 f(y)}{\partial y^2} + \alpha_m^4 f(y) \right] = 0 \quad (38) \end{aligned}$$

The final result shows that the variables can be separate. However, we still do not know what the y function is. If Eq. 38 is divided by $\sin(\alpha_m x)$, the result is:

$$\frac{\partial^4 f(y)}{\partial y^4} - 2 \alpha_m^2 \frac{\partial^2 f(y)}{\partial y^2} + \alpha_m^4 f(y) = 0 \quad (39)$$

Solving Eq. 39 will give rise to a similar solution as Eq. 35. Using one of the possible solutions, the y function can be expressed as:

$$f(y) = C_1 e^{\alpha_m y} + C_2 e^{-\alpha_m y} + C_3 y e^{\alpha_m y} + C_4 y e^{-\alpha_m y} \quad (40)$$

The overall expression for Z would then be:

$$Z = \sin(\alpha_m x) [C_1 e^{\alpha_m y} + C_2 e^{-\alpha_m y} + C_3 y e^{\alpha_m y} + C_4 y e^{-\alpha_m y}] \quad (41)$$

Now that there is an expression for Z , exact values have to be determined for the constants. This procedure is detailed in the following section.

Using Boundary Conditions to Solve for Constants

There are known solutions to the biharmonic equation which can be used directly. The previous two sections highlighted methods as to how these solutions can be obtained. The next step requires the use of boundary conditions to solve for the constants followed by the functions implementation. The series solution, shown below, is another example of an existing solution that can be used to solve this problem.

$$Z(x, y) = \sum_{m=0}^{\alpha} \sum_{n=0}^{\beta} C_{mn} x^m y^n \quad (42)$$

Where α and β are the highest orders to which the x and y variables exhibit respectively. Then only thing that would be needed to use this solution are values of C_{mn} .

For the case of a uniaxial compression in the x direction, a modified form of the biharmonic equation is required. This function is shown below.

$$\nabla^4 Z = \frac{1}{D} \left(N_x \frac{\partial^2 Z}{\partial x^2} + N_y \frac{\partial^2 Z}{\partial y^2} + 2N_{xy} \frac{\partial^2 Z}{\partial x \partial y} \right) \quad (43)$$

Since the only force acting is in the x direction, N_y and N_{xy} are set equal to zero. Also, because the force is a compression force, N_x should undergo a sign change. Hence the expression becomes:

$$\nabla^4 Z = -\frac{N_x}{D} \left(\frac{\partial^2 Z}{\partial x^2} \right) \quad (44)$$

Due to this modification, the y function will take on a different form.

$$f(y) = C_1 e^{-\alpha y} + C_2 e^{\alpha y} + C_3 \cos(\beta y) + C_4 \sin(\beta y) \quad (45)$$

$$\alpha_m = \frac{m\pi}{W}; \quad \alpha = \sqrt{\alpha_m^2 + \sqrt{\frac{N_x}{D}} \alpha_m^2}; \quad \beta = \sqrt{-\alpha_m^2 + \sqrt{\frac{N_x}{D}} \alpha_m^2}$$

Hence the expression for Z will be:

$$Z = \sin(\alpha_m x) [C_1 e^{-\alpha y} + C_2 e^{\alpha y} + C_3 \cos \beta y + C_4 \sin \beta y] \quad (46)$$

The boundary conditions from Eqs. 9, 10, 18 & 19 provide four independent equations for the four unknown constants. These boundary conditions as well as the expressions they provide are shown below.

When $y = 0$, $Z = 0$

$$C_1 + C_2 + C_3 = 0 \quad (47)$$

When $y = 0$, $\frac{\partial Z}{\partial y} = 0$

$$-\alpha C_1 + \alpha C_2 + \beta C_4 = 0 \quad (48)$$

When $y = L$, $\frac{\partial^2 Z}{\partial y^2} + \nu \frac{\partial^2 Z}{\partial x^2} = 0$

$$\begin{aligned} &(\alpha^2 e^{-\alpha L} - \alpha_m^2 \nu e^{-\alpha L})C_1 + (\alpha^2 e^{\alpha L} - \alpha_m^2 \nu e^{\alpha L})C_2 + (-\beta^2 \cos(\beta L) - \alpha_m^2 \nu \cos(\beta L))C_3 \\ &+ (-\beta^2 \sin(\beta L) - \alpha_m^2 \nu \sin(\beta L))C_4 = 0 \quad (49) \end{aligned}$$

When $y = L$, $\frac{\partial^3 Z}{\partial y^3} + (2 - \nu) \frac{\partial^3 Z}{\partial x^2 \partial y} = 0$

$$\begin{aligned} &(-\alpha^3 e^{-\alpha L} + (2 - \nu)\alpha_m^2 \alpha e^{-\alpha L})C_1 + (\alpha^3 e^{\alpha L} - (2 - \nu)\alpha_m^2 \alpha e^{\alpha L})C_2 \\ &+ (\beta^3 \sin(\beta L) + (2 - \nu)\alpha_m^2 \beta \sin(\beta L))C_3 \\ &+ (-\beta^3 \cos(\beta L) - (2 - \nu)\alpha_m^2 \beta \cos(\beta L))C_4 = 0 \quad (50) \end{aligned}$$

These equations are linear and hence can be solved using a linear solver. The coefficient matrix used for this set of equations is shown below.

$$\begin{bmatrix} 1 & 1 \\ -\alpha & \alpha \\ \alpha^2 e^{-\alpha L} - \alpha_m^2 \nu e^{-\alpha L} & \alpha^2 e^{\alpha L} - \alpha_m^2 \nu e^{\alpha L} \\ -\alpha^3 e^{-\alpha L} + (2-\nu)\alpha_m^2 \alpha e^{-\alpha L} & \alpha^3 e^{\alpha L} - (2-\nu)\alpha_m^2 \alpha e^{\alpha L} \\ 1 & 0 \\ 0 & \beta \\ -\beta^2 \cos \beta L - \alpha_m^2 \nu \cos \beta L & -\beta^2 \sin \beta L - \alpha_m^2 \nu \sin \beta L \\ \beta^3 \sin \beta L + (2-\nu)\alpha_m^2 \beta \sin \beta L & -\beta^3 \cos \beta L - (2-\nu)\alpha_m^2 \beta \cos \beta L \end{bmatrix}$$

The solution to a buckling problem requires that the determinant of the coefficient matrix for this set of equations to have a determinant of zero [2, 4, 9]. This means that the matrix cannot be inverted since it will be a singular matrix. Therefore matrix inversion will not be capable of arriving at a solution. However, the matrix can be modified to reduced row echelon form which will then leave four simplified equations. The fourth equation does not provide any relevant information so there are only three useful equations remaining. Because of this, any nonzero value for the fourth constant can be used and the rest of the equations can then be satisfied. The other three constants will be factors of the fourth constant because of this.

The expression for the determinant of this system is:

$$\begin{bmatrix} \alpha \left((-\beta^2 \cos \beta L - \alpha_m^2 \nu \cos \beta L)(-\beta^3 \cos \beta L - (2-\nu)\alpha_m^2 \beta \cos \beta L) - \right. \\ \left. (\beta^3 \sin \beta L + (2-\nu)\alpha_m^2 \beta \sin \beta L)(-\beta^2 \sin \beta L - \alpha_m^2 \nu \sin \beta L) \right) \\ + \beta \left((\alpha^2 e^{\alpha L} - \alpha_m^2 \nu e^{\alpha L})(\beta^3 \sin \beta L + (2-\nu)\alpha_m^2 \beta \sin \beta L) - \right. \\ \left. (-\beta^2 \cos \beta L - \alpha_m^2 \nu \cos \beta L)(\alpha^3 e^{\alpha L} - (2-\nu)\alpha_m^2 \alpha e^{\alpha L}) \right) \end{bmatrix}$$

$$\begin{aligned}
& \left[\begin{array}{l} -\alpha \left(\begin{array}{l} (-\beta^2 \cos \beta L - \alpha_m^2 \nu \cos \beta L)(-\beta^3 \cos \beta L - (2-\nu)\alpha_m^2 \beta \cos \beta L) - \\ (\beta^3 \sin \beta L + (2-\nu)\alpha_m^2 \beta \sin \beta L)(-\beta^2 \sin \beta L - \alpha_m^2 \nu \sin \beta L) \end{array} \right) \\ +\beta \left(\begin{array}{l} (\alpha^2 e^{-\alpha L} - \alpha_m^2 \nu e^{-\alpha L})(\beta^3 \sin \beta L + (2-\nu)\alpha_m^2 \beta \sin \beta L) - \\ (-\beta^2 \cos \beta L - \alpha_m^2 \nu \cos \beta L)(-\alpha^3 e^{-\alpha L} + (2-\nu)\alpha_m^2 \alpha e^{-\alpha L}) \end{array} \right) \end{array} \right] + \\
& \left[\begin{array}{l} -\alpha \left(\begin{array}{l} (\alpha^2 e^{\alpha L} - \alpha_m^2 \nu e^{\alpha L})(-\beta^3 \cos \beta L - (2-\nu)\alpha_m^2 \beta \cos \beta L) - \\ (-\beta^2 \sin \beta L - \alpha_m^2 \nu \sin \beta L)(\alpha^3 e^{\alpha L} - (2-\nu)\alpha_m^2 \alpha e^{\alpha L}) \end{array} \right) \\ -\alpha \left(\begin{array}{l} (\alpha^2 e^{-\alpha L} - \alpha_m^2 \nu e^{-\alpha L})(-\beta^3 \cos \beta L - (2-\nu)\alpha_m^2 \beta \cos \beta L) - \\ (-\beta^2 \sin \beta L - \alpha_m^2 \nu \sin \beta L)(-\alpha^3 e^{-\alpha L} + (2-\nu)\alpha_m^2 \alpha e^{-\alpha L}) \end{array} \right) \\ +\beta \left(\begin{array}{l} (\alpha^2 e^{-\alpha L} - \alpha_m^2 \nu e^{-\alpha L})(\alpha^3 e^{\alpha L} - (2-\nu)\alpha_m^2 \alpha e^{\alpha L}) - \\ (\alpha^2 e^{\alpha L} - \alpha_m^2 \nu e^{\alpha L})(-\alpha^3 e^{-\alpha L} + (2-\nu)\alpha_m^2 \alpha e^{-\alpha L}) \end{array} \right) \end{array} \right] \quad (51)
\end{aligned}$$

Once values have been selected that give rise to a determinant equal to zero, using Eq. 51, the value of the constants in the buckling equation can be obtained. With the function well defined, an analysis can now be done on the variables that are present within the equation. This analysis will give rise to critical values and show where the threshold of stability lies due to a change in certain function parameter values.

Determination of Critical Buckling Values

A key aspect of buckling analysis is the amount of stress a body can endure before it goes unstable and buckles. This analysis provides an insight into the effect that certain material properties have on the material's stability. By varying properties like the hydrogel's length, width, modulus of rigidity, Poisson ratio, and magnitude of compression, the effect of these changes on buckling can be observed. Likewise, the critical values of these parameters can also be determined. The critical value of a parameter represents a point where an increase or decrease from this point will cause instability. Whether that change should be an increase or a decrease depends on whether the plot of the parameter is a minimum or maximum curve. The methods required to obtain these values are detailed in the following sections.

Failure of Plane Strain Assumption

The plane strain assumption implies that there is no reason for there to be displacement in any direction other than those within the plane being subjected to stress[5, 7]. For example, if a material is being stressed in the x and y directions, then there should only be displacement in those directions. However, if the magnitude of the applied force increases, there is a point at which the plane strain assumption ceases to apply. This is when buckling occurs. This is due to an instability in the solution of the biharmonic equation which occurs when the determinant of the coefficient matrix for the equation's constants equals zero. From this particular moment, a lot of information can be

determined. Various parameters within the equation can be changed and their effect on the buckling solution can be observed.

Determinant Method for Critical Buckling Values

Using the analytical solution to the biharmonic equation, the magnitude of the critical force or pressure required for buckling can be determined. Similar work has been done which verifies that these quantities can be obtained[2].

Eq. 51 is the expression that represents the determinant for the solution to the constants in Eq. 46. As mentioned before, setting this determinant to zero will be representative of the instability and the buckling phenomenon. Once this is done, it is desired to determine how much compression force is necessary to cause different buckling modes to occur.

So, using the same determinant expression, the value of m was changed.

Simultaneously, with each change of m , the minimum value of N_x was solved for. It is crucial that the value of N_x be the lowest possible solution to Eq. 51. Fig. 4 shows that there are multiple solutions that exist that will result in a determinant equal to zero. Since it is the goal to obtain the critical value of N_x , we have to observe what the smallest possible value is that will cause buckling to occur.

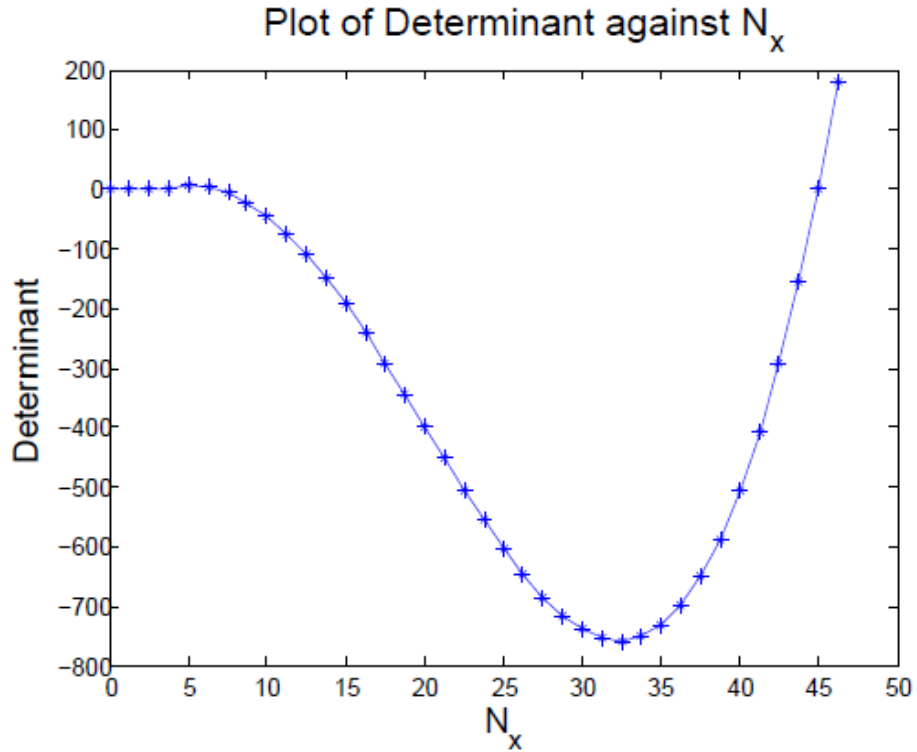


Figure 4: Determinant Behavior w.r.t. N_x

The result of this procedure gives rise to a plot like the one shown in Fig. 5. Clearly there is a minimum value of N_x and a corresponding value of m to go with it. This minimum point signifies the minimum amount of compressive force required to buckle. While the value of m at that point represents the number of half waves that would be expected in the buckled conformation.

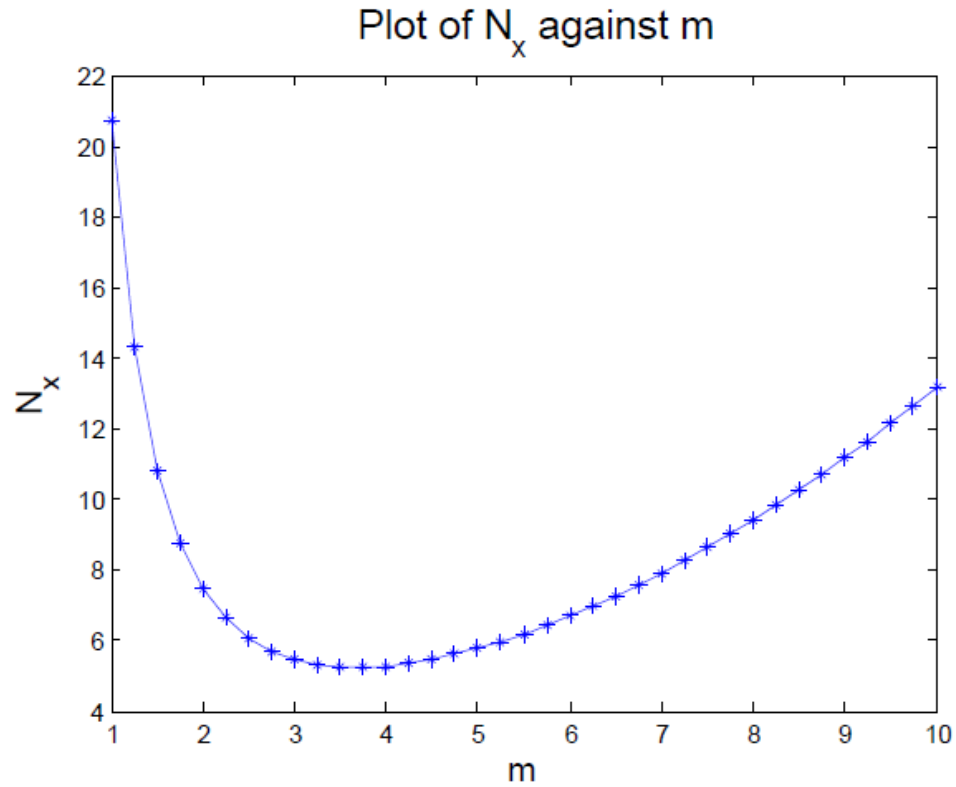


Figure 5: Plot of Solution of Critical Values of N_x

This procedure can be further expanded to determine the critical values of other parameters. However, just like the compression force, the determinant must be analyzed so that the minimum value of the parameter is used and not just any value that satisfies the determinant criteria.

The finite difference method could also be used to determine the critical values. A nonzero solution to the deflection surface would also come about when the determinant of the coefficient matrix is equal to zero. The only difference is the number of equations that will have to be used in order to solve for the values. For the sake of simplicity and ease of programming, this procedure was not used.

Energy Method

The forces of compression acting on the hydrogel perform work on the hydrogel. Because of this there is a certain amount of work that once exceeded, will give rise to an instability and cause the onset of buckling. This energy is called the strain energy. It is defined below with the following equation[9]:

$$U = \frac{D}{2} \int \int \left\{ \left(\frac{\partial^2 Z}{\partial x^2} + \frac{\partial^2 Z}{\partial y^2} \right)^2 - 2(1 - \nu) \left[\frac{\partial^2 Z}{\partial x^2} \frac{\partial^2 Z}{\partial y^2} - \left(\frac{\partial^2 Z}{\partial x \partial y} \right)^2 \right] \right\} dx dy \quad (52)$$

There will then be a minimum value of the strain energy that will give rise to a critical amount of buckling. What this amount is depends on how much work is done. The equation for the work done on the hydrogel is shown below:

$$W = -\frac{1}{2} \int \int \left[N_x \left(\frac{\partial Z}{\partial x} \right)^2 + N_y \left(\frac{\partial Z}{\partial y} \right)^2 + 2N_{xy} \frac{\partial Z}{\partial x} \frac{\partial Z}{\partial y} \right] dx dy \quad (53)$$

Since there is only compression in the x direction,

$$W = -\frac{1}{2} \int \int \left[N_x \left(\frac{\partial Z}{\partial x} \right)^2 \right] dx dy \quad (54)$$

To determine what the critical magnitude of compression is, both of the work and the strain energy have to be equal at a minimum value of the strain energy. So setting the work and strain energy equal to one another:

$$U = W$$

$$\begin{aligned} & \frac{D}{2} \iint \left\{ \left(\frac{\partial^2 Z}{\partial x^2} + \frac{\partial^2 Z}{\partial y^2} \right)^2 - 2(1-\nu) \left[\frac{\partial^2 Z}{\partial x^2} \frac{\partial^2 Z}{\partial y^2} - \left(\frac{\partial^2 Z}{\partial x \partial y} \right)^2 \right] \right\} dx dy \\ & = -\frac{1}{2} \iint \left[N_x \left(\frac{\partial Z}{\partial x} \right)^2 \right] dx dy \quad (55) \end{aligned}$$

Therefore:

$$\begin{aligned} & \frac{D}{2} \iint \left\{ \left(\frac{\partial^2 Z}{\partial x^2} + \frac{\partial^2 Z}{\partial y^2} \right)^2 - 2(1-\nu) \left[\frac{\partial^2 Z}{\partial x^2} \frac{\partial^2 Z}{\partial y^2} - \left(\frac{\partial^2 Z}{\partial x \partial y} \right)^2 \right] \right\} dx dy + \frac{1}{2} \iint \left[N_x \left(\frac{\partial Z}{\partial x} \right)^2 \right] dx dy \\ & = 0 \quad (56) \end{aligned}$$

The solutions to the work and strain energy equations are very complicated. The solutions were obtained using the Symbolic Math Toolbox in MATLAB. Since an analytical solution exists for Z, this was used in the equations and the integrals as well as the derivatives were then evaluated. The integrals were also evaluated within their limits of zero to W and zero to L for x and y respectively.

Eq. 56 can be used to solve for the necessary critical values in the same manner as with Eq. 51. The value of m can be changed and the desired critical value can be solved for. Fig. 6 shows a plot of the work and strain energy coming together at a particular point. That point is the critical value of N_x when m is one and this value is in agreement with the value calculated through the determinant method.

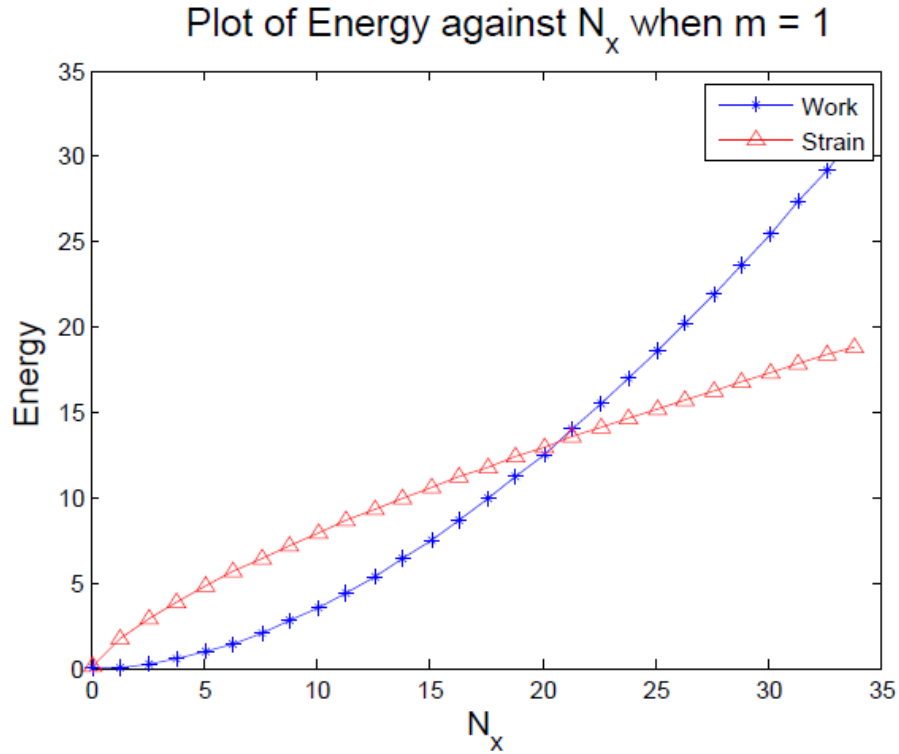


Figure 6: Work and Strain Energy Plots Used for Critical Values of N_x

This method can also be used to determine the critical values of other parameters. Due to the length and complexity of the programming code, the determinant method was the preferred method of solution.

Bending and Wasted Modes of Buckling

When a swollen hydrogel buckles it has been observed that this can occur in one of two ways[2, 15]. These types of buckling can be called bending and wasted modes of buckling. Both of these modes are shown in Fig. 7.

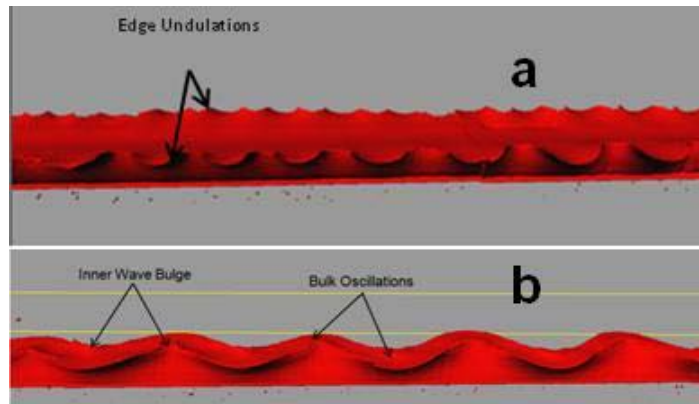


Figure 7: a) Wasted and b) Bending Modes of Buckling

The reason for this difference in buckling behavior is not well understood. Hence it is desired to possibly find out why this difference occurs as well as what factors affect it.

Equations Involved with Different Buckling Modes

An analysis exists which can describe the bending and wasted buckling modes. This analysis is similar to what was done with the analytical solution, however there are some differences. This analysis defines the flat plate or strip being in the x and z plane. The compression is still applied in the x direction and all other characteristics are the same.

Looking at the problem from this perspective will show a view from the top of how the hydrogel buckles. The equations required for this analysis are shown below.

The equations used in the biharmonic equation for a compressible and incompressible strip respectively are:

$$\chi = \cos(\alpha_m x) f(z) \quad (57)$$

$$\chi = \sin(\alpha_m x) f(z) \quad (58)$$

The overall governing equation that applies to compressible and incompressible strips for the z function is:

$$\left[\frac{d^2}{dz^2} - (\alpha_m \eta_2)^2 \right] \left[\frac{d^2}{dz^2} - (\alpha_m \eta_3)^2 \right] f(z) = 0$$

$$\frac{d^4 f(z)}{dz^4} - (\alpha_m \eta_2)^2 \frac{d^2 f(z)}{dz^2} - (\alpha_m \eta_3)^2 \frac{d^2 f(z)}{dz^2} + (\alpha_m \eta_2)^2 (\alpha_m \eta_3)^2 f(z) = 0 \quad (59)$$

If $\eta_2 \neq \eta_3$, then the equations for the bending and wasted modes of buckling respectively are:

$$f(z) = A_1 \cosh(\alpha_m \eta_2) z + A_2 \cosh(\alpha_m \eta_3) z \quad (60)$$

$$f(z) = A_1 \sinh(\alpha_m \eta_2) z + A_2 \sinh(\alpha_m \eta_3) z \quad (61)$$

If $\eta_2 = \eta_3$, then the equations for the bending and wasted modes of buckling respectively are:

$$f(z) = A_1 \cosh(\alpha_m \eta_2) z + A_2 \sinh(\alpha_m \eta_3) z \quad (62)$$

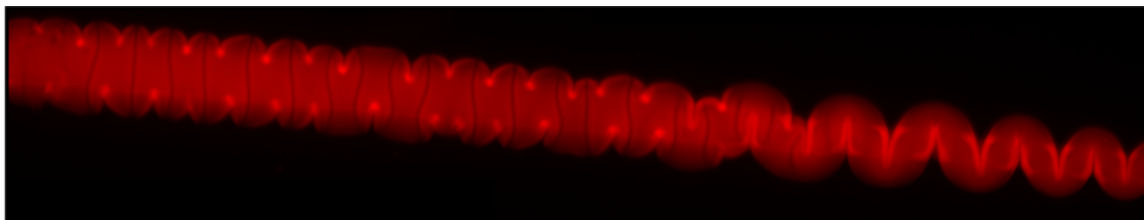
$$f(z) = A_1 \sinh(\alpha_m \eta_2) z + A_2 \cosh(\alpha_m \eta_3) z \quad (63)$$

Similar to what was highlighted before, the method for solving the constants in this equation require the use of the boundary conditions. With the boundary conditions expressed, a set of linear equations exist which can hence be solved. The instability results when the determinant of the coefficient matrix for this problem equal zero.

Unfortunately this analysis tells us nothing about why these phenomena occur. All that is presented are equations that mimic both phenomena separately. In my opinion, one equation should be capable of showing how there could be a transition from one type of buckling to the other based on certain property changes within the hydrogel strip.

Possible Reasons for Different Buckling Modes

It has also been observed that the dimensions of the confined hydrogel play a distinct role in determining which of these modes occur[2]. When the hydrogel's thickness in the z direction increases, there is an overall decrease in the amount of buckling that is observed. This could be due to a number of reasons but one of them seems to be very reasonable.



Optical 250X

Figure 8: Buckling Transition from One Mode to Another

Observe the hydrogel in Fig. 8. As the hydrogel is thickened, its volume is increased assuming that it is still the same height. Yet since it is still a part of the same hydrogel,

when it is swollen, it undergoes the same magnitude of compression as the rest of the gel. However, the part of the gel with a greater thickness will undergo less pressure while the thinner part will undergo more. The greatest magnitude of buckling is observed at the free edge which is farthest away from the confined surface. This causes a gradient in buckling which would then be a function of the thickness or the amount of pressure that that region experiences uniaxially.

The magnitude of compression in the z direction may also increase as the hydrogel gets thicker. To analyze this theory, a more complicated analysis is necessary which will require three dimensional analysis of the buckling problem. While the theoretical framework exists for three dimensional analysis of buckling, substantial modifications would have to be made to fit the theory to this problem.

Results

With all of the mentioned methods of solving the buckling problem, the goal is to determine approximately how much buckling can be expected for a specific size of confined polymer. Also, noting how changing different properties of the hydrogel can give rise to different magnitudes or types of buckling.

Finite Difference Solution

The finite difference solution is only capable of arriving at an approximation of the buckled deflection surface of the hydrogel. After using the appropriate boundary conditions to describe the gel, the approximation seems to only be able to mimic the edge undulation or the wasted form of the buckling. Curvature was induced on the free edge and the boundary conditions described in Eqs. 9,10,14,15,18 and 19 were used as well.

Combinations of forward, backward, and central finite difference schemes were used to define the coefficient matrix. Trial and error methods were used to determine the combinations of finite differences that made the system most stable. A spy of the coefficient matrix is shown in Fig. 9. The dots represent the places in the matrix where there are values. This coefficient matrix is solving for a total of one hundred grid points arranged in a five by twenty grid.

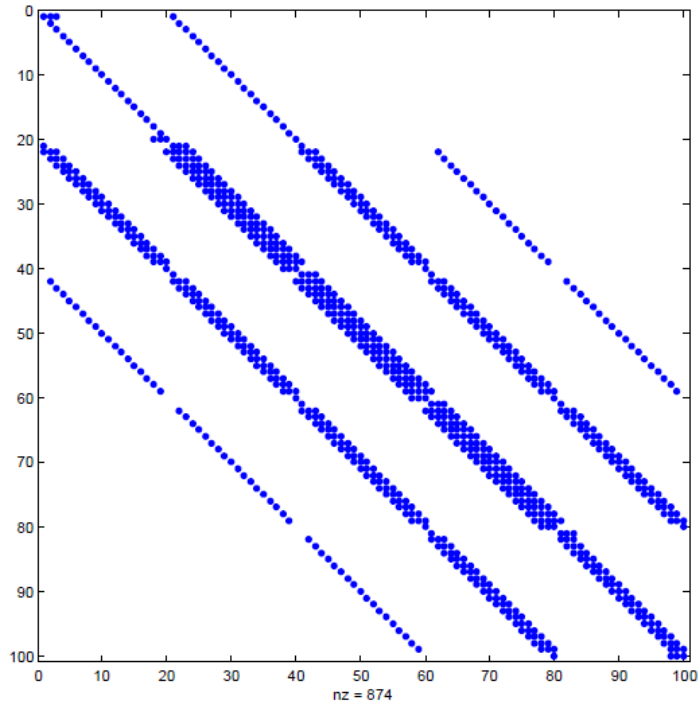


Figure 9: Spy of the Coefficient Matrix of Finite Difference Problem

Due to the fact that the stability of the finite difference solution is of great importance, limitations exist in regards to the number of grid points that can be used. Fig. 10 shows how changes in the number of grid points in the y direction affect the solution. The grid points were increased from four, to five, and then to six. When there are more than five grid points, the solution is no longer stable.

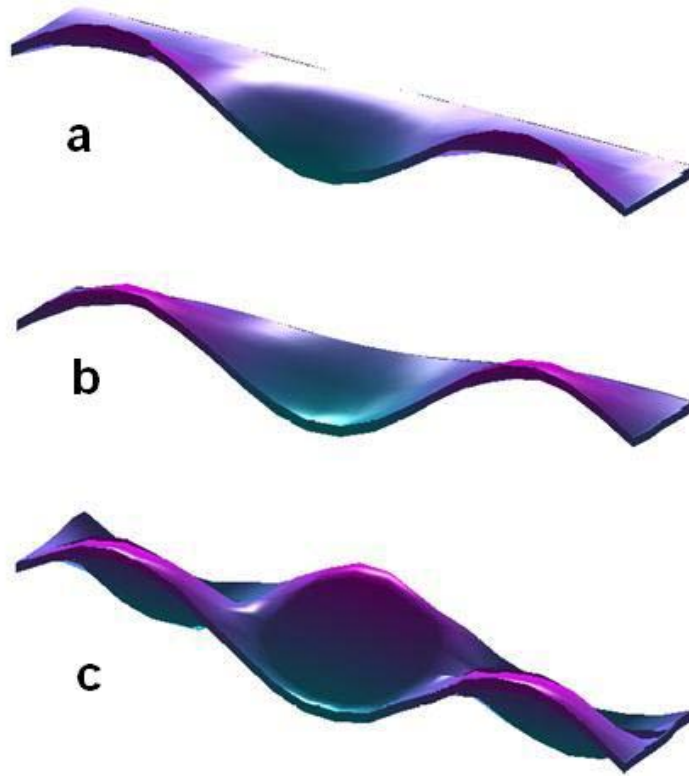


Figure 10: Solution Instabilities with Different Numbers of Grid Points on the y Axis. Twenty grid points were used on the x Axis and a) four b) five and c) six points were used on the y axis.

Fig. 11 shows how the number of grid points on the x direction affect the stability of the solution. The grid points were increased from twenty, to thirty, and then to thirty four. Thirty grid points in this direction seem appropriate. While thirty two grid points seem to be the threshold of stability, there is no need to perform the unnecessary calculations as no new information is gained by doing this.

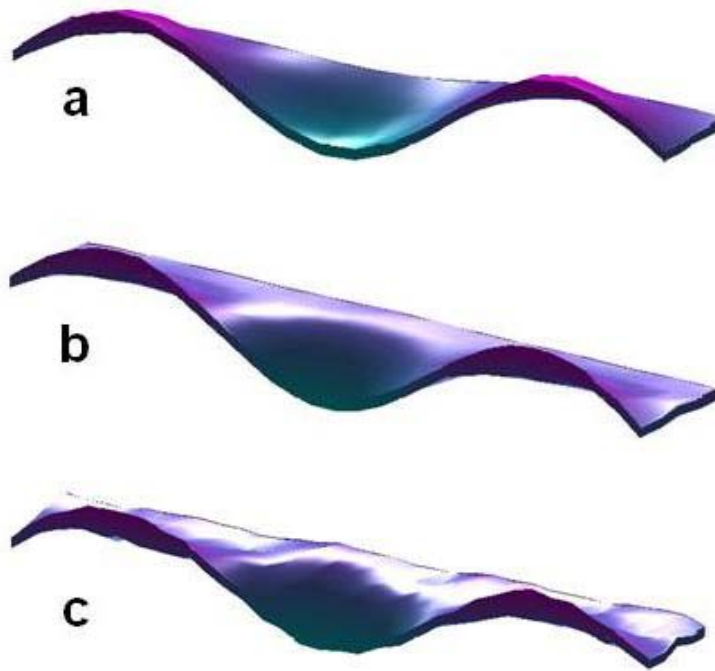


Figure 11: Solution Instabilities with Different Numbers of Grid Points on the x Axis. Five grid points were used on the y Axis and a) twenty b) thirty and c) thirty four were used on the x axis.

Fig. 11b shows the most stable solution to the problem with the greatest amount of grid points allowed. There are thirty grid points in the x direction and five in the y direction.

Analytical Solution

The analytical solution was used to perform multiple tasks. It was first used to verify the buckled deflection surface predicted by the finite difference approximation. Then, the critical values of the wavelength and compressive stress were determined so that relationships between the parameters in the equation could be found.

Fig. 12 shows the analytical solution of the buckled deflection surface.

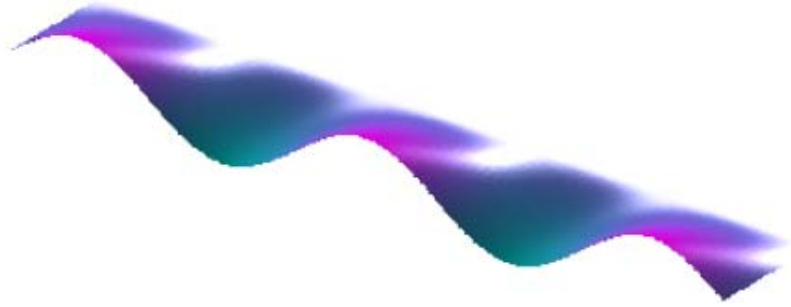


Figure 12: An Analytical Solution with Five Half Waves

Upon changing the parameters in the analytical equation, there was no noticeable change in the type of buckling that was observed. For this reason, the analytical solution as it is seems to only be capable of mimicking the bending mode of buckling just like the finite difference solution.

Determinants

The analysis of the behavior of the determinants with respect to each parameter is very necessary. If changes in the parameter do not cause the determinant to equal zero, then that parameter cannot cause buckling to occur. This can also show ahead of time what can be expected for the critical buckling values.

Fig. 13 shows a plot of how the determinant behaves using the determinant method to solve for the critical values of N_x . There are two nonzero solutions that satisfy the determinant criteria for every value of m . The minimum value of the solution was selected since this value would represent the minimum possible amount of compression required for buckling.

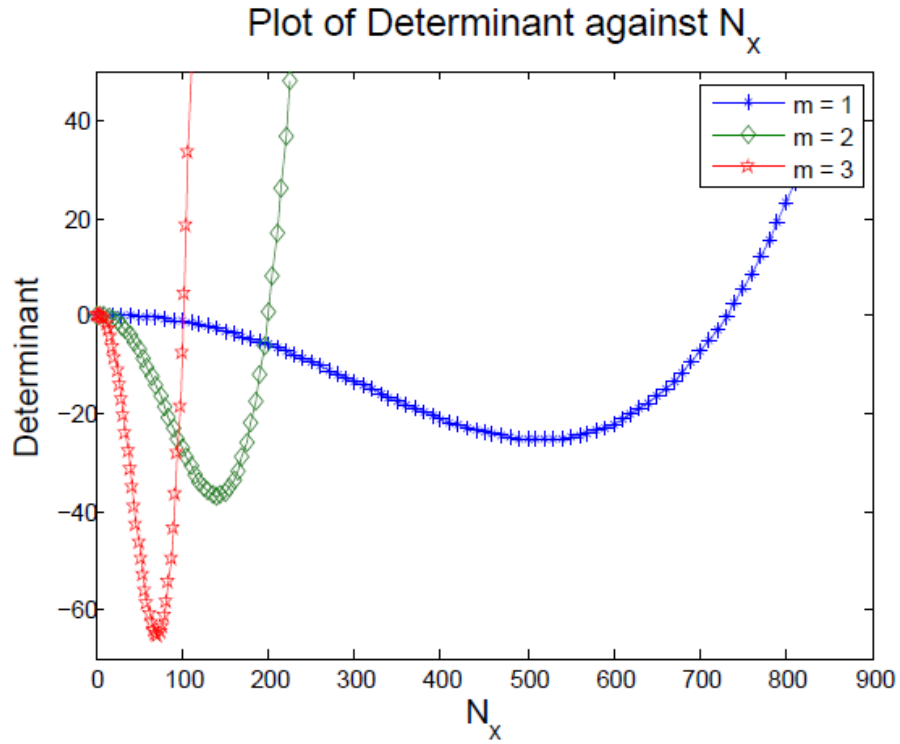


Figure 13: Determinant Behavior w.r.t. N_x at Different Values of m

Fig. 14 shows the effect that the flexural rigidity has on the determinant. Unlike any of the other parameter analyses, the flexural rigidity has only one solution that satisfies the determinant criteria.

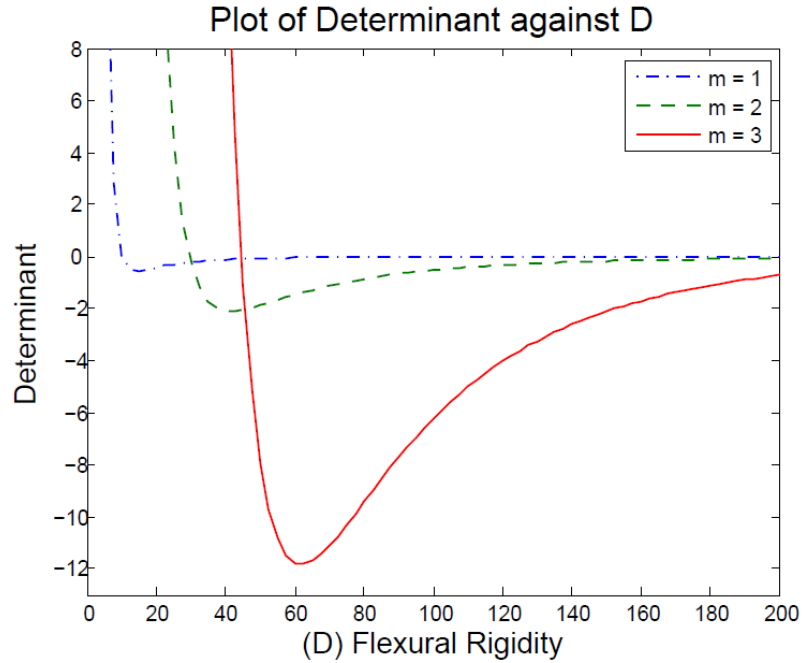


Figure 14: Determinant Behavior w.r.t. Flexural Rigidity at Different Values of m

Fig. 15 shows how the ratio of length to width affects the determinant. Upon further analysis, it was observed that there are infinite solutions that satisfy the determinant criteria with the length and width ratio. As the length to width ratio is increased, the behavior of the determinant becomes more and more unstable. Increasing the value of the length to width ratio causes the values of the determinant to deviate further to higher amplitudes in an unstable sine curve way.

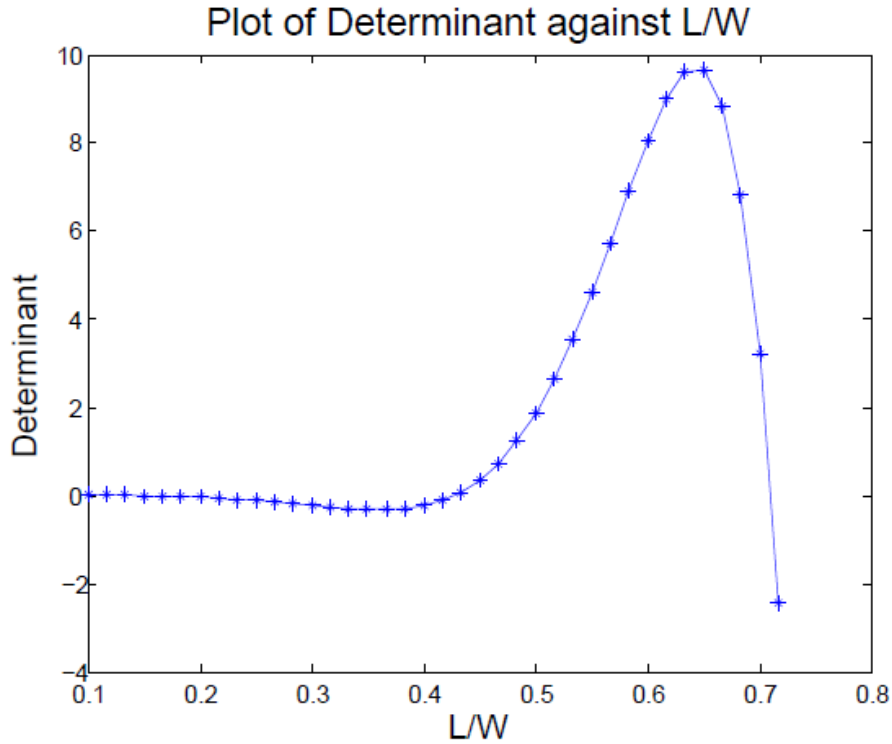


Figure 15: Determinant Behavior w.r.t. Length and Width Ratio

Analysis of the Poisson ratio shows that this parameter can have an effect on the buckling behavior. However the points at which the solutions occur are out of range for an elastic material. An elastic material's Poisson ratio is between zero and one and there were very few solutions that fell within this range.

Critical Values

All of the determinant values for the respective parameters highlighted in the previous section were solved for at different values of the wavenumber. The critical value plots all have a unique characteristic. There is either a minimum or a maximum point which highlights a limit for that parameter. What that limit means or represents depends on the parameter being analyzed.

Fig. 16 shows the critical values of flexural rigidity obtained at different values of N_x .

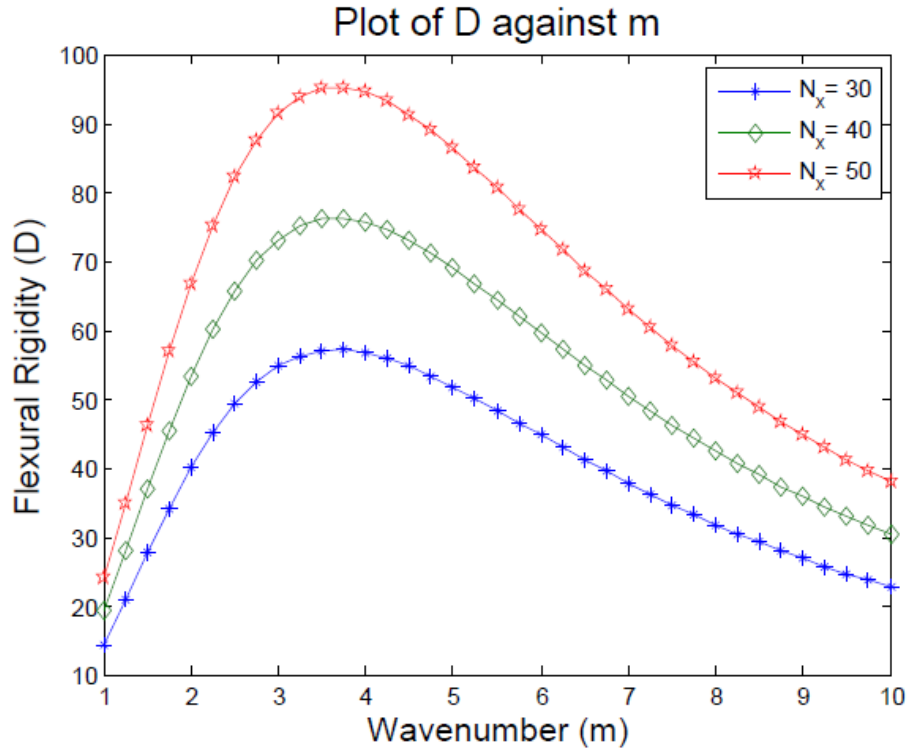


Figure 16: Critical Values of Flexural Rigidity at Different Values of m and N_x

All of the critical flexural rigidity curves have a maximum value. This maximum is the largest possible value of flexural rigidity that can cause buckling. This, in theory, does make sense. If an object is more rigid, it is less likely to buckle. So there should be an upper limit of this value, above which no buckling will occur. Increasing the magnitude of compression increases this maximum value. This also makes sense because a larger force of compression will be able to counter the effects of a more rigid body or a greater value of flexural rigidity.

Fig. 17 shows the critical values of N_x obtained at different values of flexural rigidity.

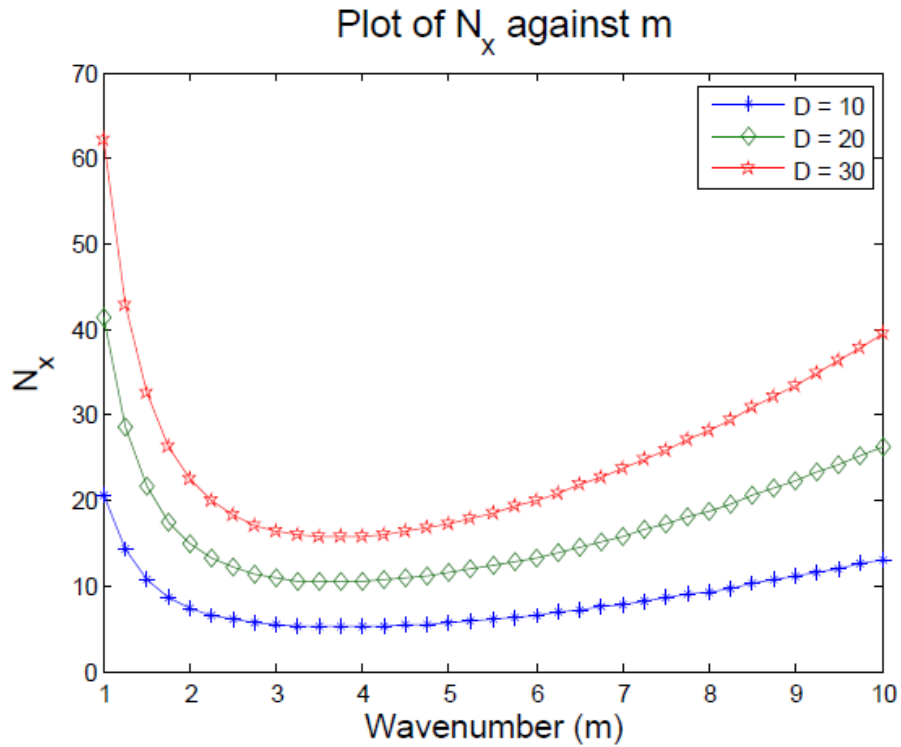


Figure 17: Critical Values of N_x at Different Values of m and Flexural Rigidity

Similar to the analysis of Fig. 16, the behavior of the critical values can be explained. An increase in the value of flexural rigidity will result in a greater force of compression necessary to cause buckling. In both Figs. 16 & 17, these changes have no effect on the wavenumber at which buckling occurs. Hence these changes have no effect on the wavelength or the number of observed buckles on the deflection surface.

Fig. 18 shows the critical values of N_x and how they are affected by changes in the length to width ratio.

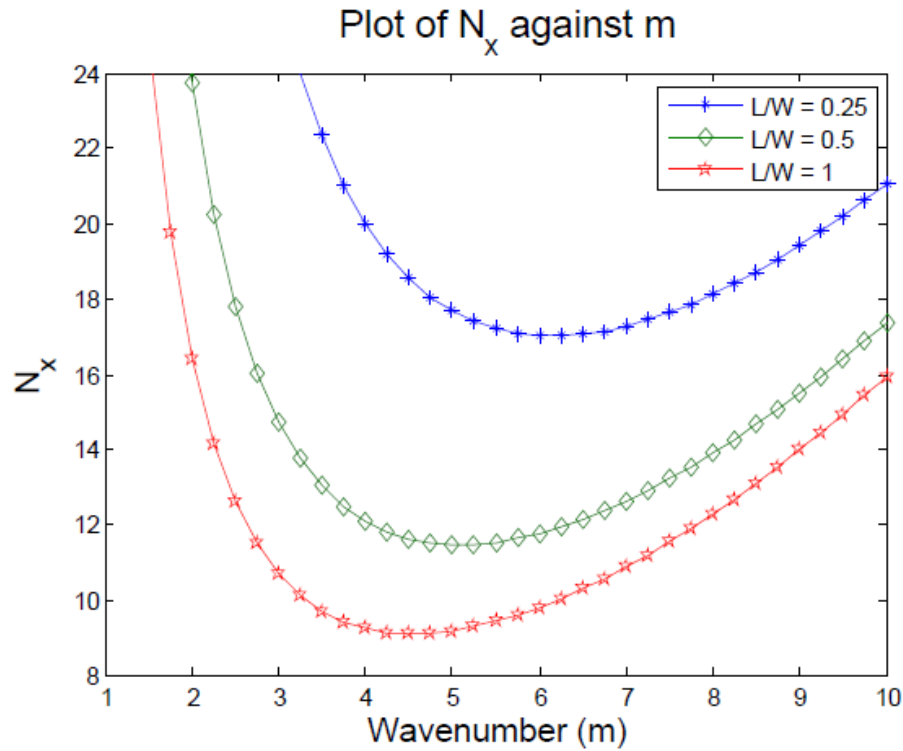


Figure 18: Critical Values of N_x at Different Values of m and Length to Width Ratio

Assuming that the length of the hydrogel is kept at a constant value, the width of the hydrogel can be changed to give rise to different values of the length to width ratio. So an increase in the width of the hydrogel will give rise to a decrease in the length to width ratio. Therefore, Fig. 18 shows that increasing the width of the hydrogel causes an increase in the minimum critical amount of compression necessary for buckling.

The changes in the length to width ratio also cause a shift in the critical buckling wavenumber. Increasing the width of the hydrogel causes the critical wavenumber to increase. This means that increasing the width of the hydrogel will cause an increase in the number of observed buckles on the deflection surface.

Fig. 19 shows the critical values of the length to width ratio and how changes in N_x affect them.

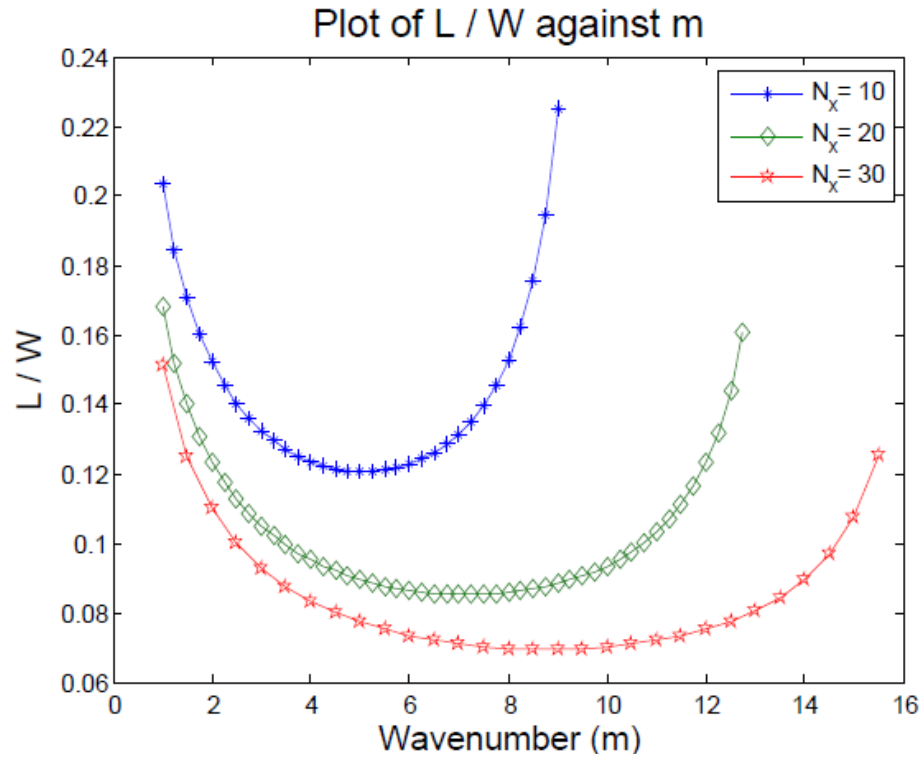


Figure 19: Critical Values of Length to Width Ratio at Different Values of m and N_x

This result is in direct correspondence to results obtained in Fig. 18. A wider hydrogel requires a greater amount of compression to cause buckling. Therefore more compression will cause a decrease in the critical ratio of length to width. The increase in compression will also cause an increase in the critical buckling wavenumber. This is equivalent to the relationship in Fig. 18 where a decrease in the length to width ratio also causes an increase in the critical buckling wavenumber.

The obtained results overall, highlight key relationships that exist between the parameters that are present in the thin plate buckling analysis. In reality however, these parameters may not necessarily be so separate and easy to control. If they can be controlled, the general trends of the observed relationships should still retain their validity.

Conclusion

The buckling phenomenon, in reality, is quite complex. However, assumptions can be made to simplify the problem and make analysis more feasible. Treating a surface confined polymer hydrogel like a thin plate subject to compression is a way to simplify the problem yet find out something about its buckling behavior. There are different methods of solving the equations governing this problem, but the desire to obtain quantities that are representative of this phenomenon requires the use of the analytical solution to the problem.

The relationships obtained from the analytical solution were able to show how the flexural rigidity, magnitude of compression, and the length to width ratio affected the buckling behavior. The length to width ratio is a parameter that is easy to control. However, being able to change the flexural rigidity and the magnitude of compression may introduce a new set of factors that would have to be analyzed.

For example, increasing the cross link density within the hydrogel may cause an increase in the value of the flexural rigidity. But this increase in cross link density may also limit the amount of water that can diffuse into the hydrogel. The amount of diffused water could very well determine the amount of compression that the hydrogel undergoes. As a result, both of these parameters, flexural rigidity and the magnitude of compression, depend on the cross link density. The relationship that exists between these quantities will have to be determined in order to have the desired control over them and hence the desired amount of buckling. Other relationships that are coupled to

a single parameter may exist and they must also be analyzed to see how they affect the key buckling parameters.

The current framework does not however solve for the amplitude of the buckled solution. This is an important factor that is necessary to describe buckling behavior. But this parameter is definitely a function of the swelling ratio of the gel before and after swelling. If the gel swells to a size substantially greater than its initial size, then the amplitude of the buckles should increase. Experiments would have to be done to determine what the nature of this relationship may be as it is not included in the thin plate assumption.

Another key analysis is the description of the bending and wasted modes of buckling. As was mentioned before, equations exist that describe the two phenomena separately. But a method of analysis must exist that shows how the transition would occur from one mode to the other. From what I have observed, the magnitude of compression in the z direction increases as the hydrogel gets thicker and this could explain the two different types of buckling. However, the appropriate equations would have to be used in order to verify that this idea has validity. Due to a lack of time, further analysis in this regard could not be pursued.

Further experiments may also have to be done to verify that the described relationships from the thin plate assumption are accurate. For example, an experiment where the length of the gel is kept constant, yet different widths of the hydrogel are used would help to verify the length to width ratio relationships.

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Appendices

Appendix A: Nomenclature

A_n, C_n, C_{mn} - Equation Constant

C - Coefficient Matrix

D - Flexural Rigidity

$e_{xx,yy,xy}$ - Strain Tensor Component

$F_{x,y,z}$ - Body Force

L - Length of Hydrogel in y Direction

m - Wavenumber (Number of Half

N_x - Uniaxial Compression Force in the x Direction

S - Solution Matrix

W - Width of hydrogel in x Direction

Z - Solution to Buckled Deflection Surface

α, α_m, β - Equation Parameters (defined in the next section)

ν - Poisson Ratio

η - Equation Parameter

$\sigma_{xx,yy,xy}$ - Stress Tensor Component

ϕ - Potential Function

λ - Equation Constant

Appendix B: Derivations of Finite Difference Equations

Starting off with the finite difference equation of a second derivative:

$$\frac{\partial^2 Z}{\partial x^2} = \frac{Z_{i+1,j} - 2Z_{i,j} + Z_{i-1,j}}{(\Delta x)^2}$$

The third order derivative is determined via the following equations:

$$\frac{\partial^3 Z}{\partial x^3} = \frac{\frac{\partial^2 Z}{\partial x^2} \Big|_{x+\frac{\Delta x}{2}} - \frac{\partial^2 Z}{\partial x^2} \Big|_{x-\frac{\Delta x}{2}}}{\Delta x}$$

$$\frac{\partial^3 Z}{\partial y^3} = \frac{\frac{\partial^2 Z}{\partial y^2} \Big|_{y+\frac{\Delta y}{2}} - \frac{\partial^2 Z}{\partial y^2} \Big|_{y-\frac{\Delta y}{2}}}{\Delta y}$$

Expressions exist for the above third order derivatives. The following derivative was derived as shown below.

$$\begin{aligned} \frac{\partial^3 Z}{\partial x^2 \partial y} &= \frac{\frac{\partial^2 Z}{\partial x^2} \Big|_{y+\frac{\Delta y}{2}} - \frac{\partial^2 Z}{\partial x^2} \Big|_{y-\frac{\Delta y}{2}}}{\Delta y} \\ &= \frac{\left(Z_{i+1,j+\frac{1}{2}} - 2Z_{i,j+\frac{1}{2}} + Z_{i-1,j+\frac{1}{2}} \right) - \left(Z_{i+1,j-\frac{1}{2}} - 2Z_{i,j-\frac{1}{2}} + Z_{i-1,j-\frac{1}{2}} \right)}{(\Delta x)^2 (\Delta y)} \end{aligned}$$

The fourth order derivative is determined via the following equations:

$$\frac{\partial^4 Z}{\partial x^4} = \frac{\frac{\partial^3 Z}{\partial x^3} \Big|_{x+\frac{\Delta x}{2}} - \frac{\partial^3 Z}{\partial x^3} \Big|_{x-\frac{\Delta x}{2}}}{\Delta x}$$

$$\frac{\partial^4 Z}{\partial y^4} = \frac{\frac{\partial^3 Z}{\partial y^3} \Big|_{y+\frac{\Delta y}{2}} - \frac{\partial^3 Z}{\partial y^3} \Big|_{y-\frac{\Delta y}{2}}}{\Delta y}$$

Expressions already exist for the first two derivatives. However, the following derivative was derived as shown below.

Appendix B Continued

$$\begin{aligned}
 \frac{\partial^4 Z}{\partial x^2 \partial y^2} &= \frac{\left. \frac{\partial^3 Z}{\partial x^2 \partial y} \right|_{y+\frac{\Delta y}{2}} - \left. \frac{\partial^3 Z}{\partial x^2 \partial y} \right|_{y-\frac{\Delta y}{2}}}{\Delta y} \\
 &= \frac{(Z_{i+1,j+1} - 2Z_{i,j+1} + Z_{i-1,j+1} - Z_{i+1,j} + 2Z_{i,j} - Z_{i-1,j})}{(\Delta x)^2 (\Delta y)^2} \\
 &\quad - \frac{(Z_{i+1,j} - 2Z_{i,j} + Z_{i-1,j} - Z_{i+1,j-1} + 2Z_{i,j-1} - Z_{i-1,j-1})}{(\Delta x)^2 (\Delta y)^2} \\
 &= \frac{Z_{i+1,j+1} - 2Z_{i,j+1} + Z_{i-1,j+1} - 2Z_{i+1,j} + 4Z_{i,j} - 2Z_{i-1,j} + Z_{i+1,j-1} - 2Z_{i,j-1} + Z_{i-1,j-1}}{(\Delta x)^2 (\Delta y)^2}
 \end{aligned}$$

Appendix C: Derivation of Biharmonic Equation in Terms of Displacement

The shearing forces are defined as:

$$Q_x = \frac{\partial M_x}{\partial x} - \frac{\partial M_{xy}}{\partial y}$$

$$Q_y = \frac{\partial M_y}{\partial y} - \frac{\partial M_{xy}}{\partial x}$$

The projection of the forces in the z direction are:

$$N_x \frac{\partial^2 Z}{\partial x^2} dxdy + \frac{\partial N_x}{\partial x} \frac{\partial Z}{\partial x} dxdy$$

$$N_y \frac{\partial^2 Z}{\partial y^2} dxdy + \frac{\partial N_y}{\partial y} \frac{\partial Z}{\partial y} dxdy$$

$$2N_{xy} \frac{\partial^2 Z}{\partial x \partial y} dxdy + \frac{\partial N_{xy}}{\partial x} \frac{\partial Z}{\partial y} dxdy + \frac{\partial N_{xy}}{\partial y} \frac{\partial Z}{\partial x} dxdy$$

The overall equilibrium expression including the addition of all the projected forces and a load (q) is:

$$\frac{\partial Q_x}{\partial x} dxdy + \frac{\partial Q_y}{\partial y} dxdy + qdxdy + N_x \frac{\partial^2 Z}{\partial x^2} dxdy + \frac{\partial N_x}{\partial x} \frac{\partial Z}{\partial x} dxdy + N_y \frac{\partial^2 Z}{\partial y^2} dxdy$$

$$+ \frac{\partial N_y}{\partial y} \frac{\partial Z}{\partial y} dxdy + 2N_{xy} \frac{\partial^2 Z}{\partial x \partial y} dxdy + \frac{\partial N_{xy}}{\partial x} \frac{\partial Z}{\partial y} dxdy + \frac{\partial N_{xy}}{\partial y} \frac{\partial Z}{\partial x} dxdy = 0$$

Or,

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + q + N_x \frac{\partial^2 Z}{\partial x^2} + \frac{\partial N_x}{\partial x} \frac{\partial Z}{\partial x} + N_y \frac{\partial^2 Z}{\partial y^2} + \frac{\partial N_y}{\partial y} \frac{\partial Z}{\partial y} + 2N_{xy} \frac{\partial^2 Z}{\partial x \partial y} + \frac{\partial N_{xy}}{\partial x} \frac{\partial Z}{\partial y} + \frac{\partial N_{xy}}{\partial y} \frac{\partial Z}{\partial x}$$

$$= 0$$

Since N_x , N_y , and N_{xy} are constants, the expression becomes:

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} = - \left(q + N_x \frac{\partial^2 Z}{\partial x^2} + N_y \frac{\partial^2 Z}{\partial y^2} + 2N_{xy} \frac{\partial^2 Z}{\partial x \partial y} \right)$$

Using the expression for the shearing forces,

$$\left(\frac{\partial^2 M_x}{\partial x^2} - \frac{\partial^2 M_{xy}}{\partial x \partial y} \right) + \left(\frac{\partial^2 M_y}{\partial y^2} - \frac{\partial^2 M_{xy}}{\partial x \partial y} \right) = - \left(q + N_x \frac{\partial^2 Z}{\partial x^2} + N_y \frac{\partial^2 Z}{\partial y^2} + 2N_{xy} \frac{\partial^2 Z}{\partial x \partial y} \right)$$

Appendix C Continued

$$\frac{\partial^2 M_x}{\partial x^2} - 2 \frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} = - \left(q + N_x \frac{\partial^2 Z}{\partial x^2} + N_y \frac{\partial^2 Z}{\partial y^2} + 2N_{xy} \frac{\partial^2 Z}{\partial x \partial y} \right)$$

The moments are defined as:

$$M_x = -D \left(\frac{\partial^2 Z}{\partial x^2} + \nu \frac{\partial^2 Z}{\partial y^2} \right)$$

$$M_y = -D \left(\frac{\partial^2 Z}{\partial y^2} + \nu \frac{\partial^2 Z}{\partial x^2} \right)$$

$$M_{xy} = D(1 - \nu) \frac{\partial^2 Z}{\partial x \partial y}$$

Substituting the moments into the expression:

$$\begin{aligned} -D \left[\frac{\partial^2}{\partial x^2} \left(\frac{\partial^2 Z}{\partial x^2} + \nu \frac{\partial^2 Z}{\partial y^2} \right) + 2(1 - \nu) \frac{\partial^2}{\partial x \partial y} \left(\frac{\partial^2 Z}{\partial x \partial y} \right) + \frac{\partial^2}{\partial y^2} \left(\frac{\partial^2 Z}{\partial y^2} + \nu \frac{\partial^2 Z}{\partial x^2} \right) \right] \\ = - \left(q + N_x \frac{\partial^2 Z}{\partial x^2} + N_y \frac{\partial^2 Z}{\partial y^2} + 2N_{xy} \frac{\partial^2 Z}{\partial x \partial y} \right) \end{aligned}$$

$$\begin{aligned} \frac{\partial^4 Z}{\partial x^4} + \nu \frac{\partial^4 Z}{\partial x^2 \partial y^2} + 2 \frac{\partial^4 Z}{\partial x^2 \partial y^2} - 2\nu \frac{\partial^4 Z}{\partial x^2 \partial y^2} + \frac{\partial^4 Z}{\partial y^4} + \nu \frac{\partial^4 Z}{\partial x^2 \partial y^2} \\ = \frac{1}{D} \left(q + N_x \frac{\partial^2 Z}{\partial x^2} + N_y \frac{\partial^2 Z}{\partial y^2} + 2N_{xy} \frac{\partial^2 Z}{\partial x \partial y} \right) \end{aligned}$$

$$\frac{\partial^4 Z}{\partial x^4} + 2 \frac{\partial^4 Z}{\partial x^2 \partial y^2} + \frac{\partial^4 Z}{\partial y^4} = \frac{1}{D} \left(q + N_x \frac{\partial^2 Z}{\partial x^2} + N_y \frac{\partial^2 Z}{\partial y^2} + 2N_{xy} \frac{\partial^2 Z}{\partial x \partial y} \right)$$

In the absence of a load and compression in directions other than the x direction, the expression simplifies to:

$$\nabla^4 Z = \frac{1}{D} \left(N_x \frac{\partial^2 Z}{\partial x^2} \right)$$

If the force acting is a compression, the sign of N_x will be negative. Otherwise there is no need for a sign change.

Appendix D: Proof of Solution to Biharmonic Equation

If the proposed solution is as follows,

$$Z = \sin(\alpha_m x) [C_1 e^{-\alpha y} + C_2 e^{\alpha y} + C_3 \cos \beta y + C_4 \sin \beta y]$$

$$\alpha_m = \frac{m\pi}{W} ; \alpha = \sqrt{\alpha_m^2 + \sqrt{\frac{N_x}{D}} \alpha_m^2} ; \beta = \sqrt{-\alpha_m^2 + \sqrt{\frac{N_x}{D}} \alpha_m^2}$$

The biharmonic equation with a compression force in the x direction is:

$$\nabla^4 Z = \frac{\partial^4 Z}{\partial x^4} + \frac{\partial^4 Z}{\partial y^4} + 2 \frac{\partial^4 Z}{\partial x^2 \partial y^2} = -\frac{N_x}{D} \frac{\partial^2 Z}{\partial x^2}$$

Solving for the LHS and dividing by $\sin(\alpha_m x)$:

$$\begin{aligned} & \alpha_m^4 [C_1 e^{-\alpha y} + C_2 e^{\alpha y} + C_3 \cos \beta y + C_4 \sin \beta y] \\ & - 2\alpha_m^2 [\alpha^2 C_1 e^{-\alpha y} + \alpha^2 C_2 e^{\alpha y} - \beta^2 C_3 \cos \beta y - \beta^2 C_4 \sin \beta y] \\ & + [\alpha^4 C_1 e^{-\alpha y} + \alpha^4 C_2 e^{\alpha y} + \beta^4 C_3 \cos \beta y + \beta^4 C_4 \sin \beta y] \end{aligned}$$

Which further simplifies to:

$$\begin{aligned} & (\alpha_m^4 - 2\alpha_m^2 \alpha^2 + \alpha^4) C_1 e^{-\alpha y} + (\alpha_m^4 - 2\alpha_m^2 \alpha^2 + \alpha^4) C_2 e^{\alpha y} \\ & + (\alpha_m^4 + 2\alpha_m^2 \beta^2 + \beta^4) C_3 \cos \beta y + (\alpha_m^4 + 2\alpha_m^2 \beta^2 + \beta^4) C_4 \sin \beta y \end{aligned}$$

Evaluating the coefficients:

$$\begin{aligned} (\alpha_m^4 - 2\alpha_m^2 \alpha^2 + \alpha^4) &= \alpha_m^4 - 2\alpha_m^2 \left(\alpha_m^2 + \sqrt{\frac{N_x}{D}} \alpha_m^2 \right) + \left(\alpha_m^2 + \sqrt{\frac{N_x}{D}} \alpha_m^2 \right)^2 \\ &= \alpha_m^4 - 2\alpha_m^4 - 2\alpha_m^2 \sqrt{\frac{N_x}{D}} \alpha_m^2 + \alpha_m^4 + 2\alpha_m^2 \sqrt{\frac{N_x}{D}} \alpha_m^2 + \frac{N_x}{D} \alpha_m^2 \\ &= \frac{N_x}{D} \alpha_m^2 \end{aligned}$$

Appendix D Continued

$$\begin{aligned}
 (\alpha_m^4 + 2\alpha_m^2\beta^2 + \beta^4) &= \alpha_m^4 + 2\alpha_m^2 \left(-\alpha_m^2 + \sqrt{\frac{N_x}{D}\alpha_m^2} \right) + \left(-\alpha_m^2 + \sqrt{\frac{N_x}{D}\alpha_m^2} \right)^2 \\
 &= \alpha_m^4 - 2\alpha_m^4 + 2\alpha_m^2 \sqrt{\frac{N_x}{D}\alpha_m^2} + \alpha_m^4 - 2\alpha_m^2 \sqrt{\frac{N_x}{D}\alpha_m^2} + \frac{N_x}{D}\alpha_m^2 \\
 &= \frac{N_x}{D}\alpha_m^2
 \end{aligned}$$

Hence the LHS is:

$$\frac{N_x}{D}\alpha_m^2 [C_1 e^{-\alpha y} + C_2 e^{\alpha y} + C_3 \cos \beta y + C_4 \sin \beta y]$$

The RHS is

$$-\frac{N_x}{D} \frac{\partial^2 Z}{\partial x^2} = -\frac{N_x}{D} [-\alpha_m^2 \sin(\alpha_m x)] [C_1 e^{-\alpha y} + C_2 e^{\alpha y} + C_3 \cos \beta y + C_4 \sin \beta y]$$

Dividing by $\sin(\alpha_m x)$ just as with the LHS, RHS then becomes:

$$\frac{N_x}{D}\alpha_m^2 [C_1 e^{-\alpha y} + C_2 e^{\alpha y} + C_3 \cos \beta y + C_4 \sin \beta y]$$

Since the left and right hand sides are equal, the proposed solution satisfies the biharmonic equation.

Appendix E: Derivation of the Strain Energy Equation

The expression for the strain energy is obtained by analyzing the contributions due to both bending and twisting. The expression for the strain energy due to bending is:

$$dU_B = -\frac{1}{2} \left(M_x \frac{\partial^2 Z}{\partial x^2} + M_y \frac{\partial^2 Z}{\partial y^2} \right) dx dy$$

The bending moments with respect to x and y are defined as:

$$M_x = -D \left(\frac{\partial^2 Z}{\partial x^2} + \nu \frac{\partial^2 Z}{\partial y^2} \right)$$

$$M_y = -D \left(\frac{\partial^2 Z}{\partial y^2} + \nu \frac{\partial^2 Z}{\partial x^2} \right)$$

Substituting the moments into the strain energy equation due to bending,

$$dU_B = -\frac{1}{2} \left(-D \left(\frac{\partial^2 Z}{\partial x^2} + \nu \frac{\partial^2 Z}{\partial y^2} \right) \frac{\partial^2 Z}{\partial x^2} - D \left(\frac{\partial^2 Z}{\partial y^2} + \nu \frac{\partial^2 Z}{\partial x^2} \right) \frac{\partial^2 Z}{\partial y^2} \right) dx dy$$

$$dU_B = \frac{D}{2} \left(\left(\frac{\partial^2 Z}{\partial x^2} \right)^2 + \nu \frac{\partial^2 Z}{\partial x^2} \frac{\partial^2 Z}{\partial y^2} + \left(\frac{\partial^2 Z}{\partial y^2} \right)^2 + \nu \frac{\partial^2 Z}{\partial x^2} \frac{\partial^2 Z}{\partial y^2} \right) dx dy$$

$$dU_B = \frac{D}{2} \left(\left(\frac{\partial^2 Z}{\partial x^2} \right)^2 + 2\nu \frac{\partial^2 Z}{\partial x^2} \frac{\partial^2 Z}{\partial y^2} + \left(\frac{\partial^2 Z}{\partial y^2} \right)^2 \right) dx dy$$

The strain energy due to twisting is defined as:

$$dU_T = M_{xy} \frac{\partial^2 Z}{\partial x \partial y} dx dy$$

The twisting moment is defined as:

$$M_{xy} = D(1 - \nu) \frac{\partial^2 Z}{\partial x \partial y}$$

Substituting the twisting moment into the strain energy due to twisting equation,

$$dU_T = D(1 - \nu) \left(\frac{\partial^2 Z}{\partial x \partial y} \right)^2 dx dy$$

Combining the strain energies due to bending and twisting,

Appendix E Continued

$$dU = dU_B + dU_T$$

$$dU = \frac{D}{2} \left(\left(\frac{\partial^2 Z}{\partial x^2} \right)^2 + 2\nu \frac{\partial^2 Z}{\partial x^2} \frac{\partial^2 Z}{\partial y^2} + \left(\frac{\partial^2 Z}{\partial y^2} \right)^2 \right) dx dy + D(1 - \nu) \left(\frac{\partial^2 Z}{\partial x \partial y} \right)^2 dx dy$$

$$dU = \frac{D}{2} \left[\left(\frac{\partial^2 Z}{\partial x^2} \right)^2 + 2\nu \frac{\partial^2 Z}{\partial x^2} \frac{\partial^2 Z}{\partial y^2} + \left(\frac{\partial^2 Z}{\partial y^2} \right)^2 + 2(1 - \nu) \left(\frac{\partial^2 Z}{\partial x \partial y} \right)^2 \right] dx dy$$

The overall strain energy is obtained by integrating the overall strain energy expression.

$$U = \frac{D}{2} \int \int \left[\left(\frac{\partial^2 Z}{\partial x^2} \right)^2 + 2\nu \frac{\partial^2 Z}{\partial x^2} \frac{\partial^2 Z}{\partial y^2} + \left(\frac{\partial^2 Z}{\partial y^2} \right)^2 + 2(1 - \nu) \left(\frac{\partial^2 Z}{\partial x \partial y} \right)^2 \right] dx dy$$

According to Timoshenko, the overall expression can be expressed as [reference]:

$$U = \frac{D}{2} \int \int \left[\left(\frac{\partial^2 Z}{\partial x^2} + \frac{\partial^2 Z}{\partial y^2} \right)^2 - 2(1 - \nu) \left(\frac{\partial^2 Z}{\partial x^2} \frac{\partial^2 Z}{\partial y^2} - \left(\frac{\partial^2 Z}{\partial x \partial y} \right)^2 \right) \right] dx dy$$

This expression is not algebraically equivalent to the other expression yet it still yields solutions that are in agreement with the determinant method.

Appendix F: Determinant of a 4x4 Matrix

If a 4x4 matrix is defined as:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

The determinant will be

$$\begin{aligned} & a_{11} \det \begin{bmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{bmatrix} - a_{12} \det \begin{bmatrix} a_{21} & a_{23} & a_{24} \\ a_{31} & a_{33} & a_{34} \\ a_{41} & a_{43} & a_{44} \end{bmatrix} + a_{13} \det \begin{bmatrix} a_{21} & a_{22} & a_{24} \\ a_{31} & a_{32} & a_{34} \\ a_{41} & a_{42} & a_{44} \end{bmatrix} \\ & - a_{14} \det \begin{bmatrix} a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{bmatrix} \end{aligned}$$

The determinants of the 3x3 matrices are as follows:

$$\det \begin{bmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{bmatrix} = a_{22}(a_{33}a_{44} - a_{34}a_{43}) - a_{23}(a_{32}a_{44} - a_{34}a_{42}) + a_{24}(a_{32}a_{43} - a_{33}a_{42})$$

$$\det \begin{bmatrix} a_{21} & a_{23} & a_{24} \\ a_{31} & a_{33} & a_{34} \\ a_{41} & a_{43} & a_{44} \end{bmatrix} = a_{21}(a_{33}a_{44} - a_{34}a_{43}) - a_{23}(a_{31}a_{44} - a_{34}a_{41}) + a_{24}(a_{31}a_{43} - a_{33}a_{41})$$

$$\det \begin{bmatrix} a_{21} & a_{22} & a_{24} \\ a_{31} & a_{32} & a_{34} \\ a_{41} & a_{42} & a_{44} \end{bmatrix} = a_{21}(a_{32}a_{44} - a_{34}a_{42}) - a_{22}(a_{31}a_{44} - a_{34}a_{41}) + a_{24}(a_{31}a_{42} - a_{32}a_{41})$$

Appendix F Continued

$$\det \begin{bmatrix} a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{bmatrix} = a_{21}(a_{32}a_{43} - a_{33}a_{42}) - a_{22}(a_{31}a_{43} - a_{33}a_{41}) + a_{23}(a_{31}a_{42} - a_{32}a_{41})$$

Hence the overall expression for the determinant is:

$$\begin{aligned} & a_{11} [a_{22}(a_{33}a_{44} - a_{34}a_{43}) - a_{23}(a_{32}a_{44} - a_{34}a_{42}) + a_{24}(a_{32}a_{43} - a_{33}a_{42})] \\ & - a_{12} [a_{21}(a_{33}a_{44} - a_{34}a_{43}) - a_{23}(a_{31}a_{44} - a_{34}a_{41}) + a_{24}(a_{31}a_{43} - a_{33}a_{41})] \\ & + a_{13} [a_{21}(a_{32}a_{44} - a_{34}a_{42}) - a_{22}(a_{31}a_{44} - a_{34}a_{41}) + a_{24}(a_{31}a_{42} - a_{32}a_{41})] \\ & - a_{14} [a_{21}(a_{32}a_{43} - a_{33}a_{42}) - a_{22}(a_{31}a_{43} - a_{33}a_{41}) + a_{23}(a_{31}a_{42} - a_{32}a_{41})] \end{aligned}$$

Appendix G: MATLAB Code

Finite Difference Approximation of Biharmonic Equation

```
function Buckling_2D
%           CENTRAL FINITE DIFFERENCE APPROXIMATION
%           OF THE BIHARMONIC EQUATION
%
%           by: Abiola Shitta
%           (Grid defined below)
%
%           Z(i,j+2)
%           Z(i-1,j+1) Z(i,j+1) Z(i+1,j+1)
%Z(i-2,j) Z(i-1,j)  Z(i,j)  Z(i+1,j)  Z(i+2,j)
%           Z(i-1,j-1) Z(i,j-1) Z(i+1,j-1)
%           Z(i,j-2)
%Used to solve:
%
%
%           d4Z      d4Z      d4Z      Nx      d2Z
%           ----- + ----- + 2----- + ----- ----- = 0
%           dx4      dy4      dx2dy2   D      dx2
clc;
clear;
close all;
%Width of Gel (x direction)%
W=30;
%Length of Gel (y direction)%
L=5;
%Thickness of Gel (z direction)%
H=0.5;

%No of Grid Points along Width%
NW=20;
%No of Grid Points along Length%
NL=5;

%Step Size along Width%
dw=W/(NW-1);
%Step Size along Length%
dl=L/(NL-1);

Nx=100;

%Poisson Ratio
pr=0.25;
%Modulus of Elasticity
E=10*(12*(1-pr^2))/(H^3);
%Flexural Rigidity
D=E*H^3/(12*(1-pr^2));
N=NL*NW;
```

Appendix G Continued

```

Z=zeros(N);
%Coefficients%
a=dw^4;
b=2*dw^2*d1^2;
c=-4*dw^4-4*dw^2*d1^2;
d=d1^4;
e=-4*d1^4-4*dw^2*d1^2+(Nx/D)*dw^2*d1^4;
f=6*d1^4+6*dw^4+8*dw^2*d1^4-2*(Nx/D)*dw^2*d1^4;
%Initial Coefficient Matrix
for i=1:N
    if i+2*NW<=N
        Z(i,i+2*(NW))=a;           %Z(i,j+2)
    end
    if i+1+NW<=N
        Z(i,i+1+(NW))=b;         %Z(i+1,j+1)
    end
    if i+NW<=N
        Z(i,i+1*(NW))=c;         %Z(i,j+1)
    end
    if i+1-NW>0
        Z(i,i+1-(NW))=b;         %Z(i+1,j-1)
    end
    if i-2>0
        Z(i,i-2)=d;              %Z(i-2,j)
    end
    if i-1>0
        Z(i,i-1)=e;              %Z(i-1,j)
    end
    Z(i,i)=f;                    %Z(i,j)
    if i+1<=N
        Z(i,i+1)=e;              %Z(i+1,j)
    end
    if i+2<=N
        Z(i,i+2)=d;              %Z(i+2,j)
    end
    if i-1+(NW)<=N
        Z(i,i-1+(NW))=b;         %Z(i-1,j+1)
    end
    if i-1*NW>0
        Z(i,i-1*(NW))=c;         %Z(i,j-1)
    end
    if i-1-NW>0
        Z(i,i-1-(NW))=b;         %Z(i-1,j-1)
    end
    if i-2*NW>0
        Z(i,i-2*(NW))=a;         %Z(i,j-2)
    end
end
%Initial Solution Matrix%
C=zeros(N,1);

%BOUNDARY CONDITIONS%
%Lower y boundary (dZ/dy-Z=0) (FFD)
for i=2:NW-1

```

Appendix G Continued

```

Z(i,:) = 0;
if i + NW <= N
    Z(i, i + 1 * (NW)) = 1;           %Z(i, j + 1)
end
Z(i, i) = -1 * dl;                   %Z(i, j)
end
%Lower x boundary (FFD)
for i = 1 : NL
    Z(NW * (i - 1) + 1, :) = 0;
    if NW * (i - 1) + 1 + 2 <= N
        Z(NW * (i - 1) + 1, NW * (i - 1) + 1 + 2) = dl ^ 2;           %Z(i + 2, j)
    end
    if NW * (i - 1) + 1 + NW <= N
        Z(NW * (i - 1) + 1, NW * (i - 1) + 1 + 1 * (NW)) = pr * dw ^ 2;           %Z(i, j + 1)
    end
    Z(NW * (i - 1) + 1, NW * (i - 1) + 1) = dl ^ 2 - 2 * pr * dw ^ 2 - dw ^ 2 * dl ^ 2;           %Z(i, j)
    if NW * (i - 1) + 1 + 1 <= N
        Z(NW * (i - 1) + 1, NW * (i - 1) + 1 + 1) = -2 * dl ^ 2;           %Z(i + 1, j)
    end
    if NW * (i - 1) + 1 - 1 * NW > 0
        Z(NW * (i - 1) + 1, NW * (i - 1) + 1 - 1 * (NW)) = pr * dw ^ 2;           %Z(i, j - 1)
    end
end
end
%Upper x boundary (BFD)
for i = 1 : NL
    Z(NW * i, :) = 0;
    if NW * i + NW <= N
        Z(NW * i, NW * i + 1 * (NW)) = pr * dw ^ 2;           %Z(i, j + 1)
    end
    if NW * i - 1 > 0
        Z(NW * i, NW * i - 1) = -2 * dl ^ 2;           %Z(i - 1, j)
    end
    if NW * i - 2 > 0
        Z(NW * i, NW * i - 2) = dl ^ 2;           %Z(i - 2, j)
    end
    Z(NW * i, NW * i) = dl ^ 2 - 2 * pr * dw ^ 2 - dw ^ 2 * dl ^ 2;           %Z(i, j)
    if NW * i - 1 * NW > 0
        Z(NW * i, NW * i - 1 * (NW)) = pr * dw ^ 2;           %Z(i, j - 1)
    end
end
end
%Upper y boundary (CFD)
for i = (NL - 1) * NW + 2 : N - 1
    Z(i, :) = 0;
    if i + 2 * NW <= N
        Z(i, i + 2 * (NW)) = dw ^ 2;           %Z(i, j + 2)
    end
    if i + NW <= N
        Z(i, i + 1 * (NW)) = 2 * dw ^ 2 - 2 * (2 - pr) * dl ^ 2 - 2 * dw ^ 2 * dl;           %Z(i, j + 1)
    end
    if i - 1 - NW > 0
        Z(i, i - 1 - (NW)) = -(2 - pr) * dl ^ 2;           %Z(i - 1, j - 1)
    end
    if i - 1 + (NW) <= N
        Z(i, i - 1 + (NW)) = (2 - pr) * dl ^ 2;           %Z(i - 1, j + 1)
    end
end

```


Appendix G Continued

```

if i-1>0
    Z(i,i-1)=-2*pr*d1^3; %Z(i-1,j)
end
Z(i,i)=-4*dw^2*d1-4*pr*d1^3; %Z(i,j)
if i+1<=N
    Z(i,i+1)=-2*pr*d1^3; %Z(i+1,j)
end
if i+1+NW<=N
    Z(i,i+1+(NW))=(2-pr)*d1^2; %Z(i+1,j+1)
end
if i+1-NW>0
    Z(i,i+1-(NW))=-(2-pr)*d1^2; %Z(i+1,j-1)
end
if i-1*NW>0
    Z(i,i-1*(NW))=2*dw^2+2*(2-pr)*d1^2-2*dw^2*d1; %Z(i,j-1)
end
if i-2*NW>0
    Z(i,i-2*(NW))=dw^2; %Z(i,j-2)
end
C(i)=Z(i,:)*[zeros((NL-1)*NW+1,1);2*sin(3*pi*(i-((NL-1)*NW+1)))/(NW-1)]*ones(NW-2,1);0];
end

%Spy on Z%
spy(Z)
%CALCULATION & REARRANGEMENT%
z=Z\C;
for j=1:NL
    for i=1:NW
        Zsol(j,i)=z((j-1)*NW+i,1);
    end
end

%PLOT OF SOLUTION%
figure;
x=0:dw:W;
y=0:d1:L;
[X,Y]=meshgrid(x,y);
hold on
%Upper Z Surface
surf(X,Y,Zsol+H/2)
%Lower Z Surface
surf(X,Y,Zsol-H/2)
%Upper Y Surface
[X1,Y1]=meshgrid(x,L);
for i=1:NW
    Zsol1(i,:)=Zsol(NL,:)+H/2-(H)*i/NW;
end
surf(X1,Y1,Zsol1)
%Lower Y Surface
[X2,Y2]=meshgrid(x,0);
for i=1:NW
    Zsol2(i,:)=Zsol(1,:)+H/2-(H)*i/NW;
end

```

Appendix G Continued

```
surf(X2,Y2,Zsol2)
%Upper X Surface
[X3,Y3]=meshgrid(W,y);
for i=1:NL
    Zsol3(:,i)=Zsol(:,NW)+H/2-(H)*i/(NL);
end
surf(X3,Y3,Zsol3)
%Lower X Surface
[X4,Y4]=meshgrid(0,y);
for i=1:NL
    Zsol4(:,i)=Zsol(:,1)+H/2-(H)*i/(NL);
end
surf(X4,Y4,Zsol4)

colormap cool
colorbar
shading interp
axis equal
axis off
zoom(1.4)
lightangle(30,30)
camorbit(-155,-72)
set(gcf,'Renderer','zbuffer')
set(findobj(gca,'type','surface'),...
    'FaceLighting','phong',...
    'AmbientStrength',0.3,'DiffuseStrength',.8,...
    'SpecularStrength',.9,'SpecularExponent',40,...
    'BackFaceLighting','unlit')
```

Determinant Behavior w.r.t. N_x

This m-file was used to determine how changing certain function values would give rise to a difference in the functions determinant. It was also used as a means to determine an appropriate initial guess for the critical magnitude of compression.

```
function AnalyticalBuckling
clc
clear all
close all
m1=1; %Wavenumber
W=30; %Width of Gel
L=5; %Length of Gel
Nx0=0;
for i=1:85
    Nx(i,1)=Nx0+10*(i-1);
D=10; %Flexural Rigidity
alpham=m1*pi/W;
```

Appendix G Continued

```
nu=0.25; %Poisson Ratio
alpha=sqrt(m1^2*pi^2/(W^2)+sqrt(Nx(i,1)/D*m1^2*pi^2/(W^2)));
beta=sqrt(-m1^2*pi^2/(W^2)+sqrt(Nx(i,1)/D*m1^2*pi^2/(W^2)));

A=[1 1 1 0; -alpha alpha 0 beta;...
   (alpha^2*exp(-alpha*L)-alphan^2*nu*exp(-alpha*L)) ...
   (alpha^2*exp(alpha*L)-alphan^2*nu*exp(alpha*L)) ...
   (-beta^2*cos(beta*L)-alphan^2*nu*cos(beta*L)) ...
   (-beta^2*sin(beta*L)-alphan^2*nu*sin(beta*L));...
   (-alpha^3*exp(-alpha*L)+alphan^2*(2-nu)*alpha*exp(-alpha*L)) ...
   (alpha^3*exp(alpha*L)-alphan^2*(2-nu)*alpha*exp(alpha*L)) ...
   (beta^3*sin(beta*L)+alphan^2*(2-nu)*beta*sin(beta*L)) ...
   (-beta^3*cos(beta*L)-alphan^2*(2-nu)*beta*cos(beta*L))];

y(i,1)=det(A);
end
m2=2;
for i=1:46
    Nx1(i,1)=Nx0+5*(i-1);
    alphan=m2*pi/W;
    alpha=sqrt(m2^2*pi^2/(W^2)+sqrt(Nx1(i,1)/D*m2^2*pi^2/(W^2)));
    beta=sqrt(-m2^2*pi^2/(W^2)+sqrt(Nx1(i,1)/D*m2^2*pi^2/(W^2)));

    A=[1 1 1 0; -alpha alpha 0 beta;...
       (alpha^2*exp(-alpha*L)-alphan^2*nu*exp(-alpha*L)) ...
       (alpha^2*exp(alpha*L)-alphan^2*nu*exp(alpha*L)) ...
       (-beta^2*cos(beta*L)-alphan^2*nu*cos(beta*L)) ...
       (-beta^2*sin(beta*L)-alphan^2*nu*sin(beta*L));...
       (-alpha^3*exp(-alpha*L)+alphan^2*(2-nu)*alpha*exp(-alpha*L)) ...
       (alpha^3*exp(alpha*L)-alphan^2*(2-nu)*alpha*exp(alpha*L)) ...
       (beta^3*sin(beta*L)+alphan^2*(2-nu)*beta*sin(beta*L)) ...
       (-beta^3*cos(beta*L)-alphan^2*(2-nu)*beta*cos(beta*L))];

    y1(i,1)=det(A);
end
m3=3;
for i=1:46
    Nx2(i,1)=Nx0+2.5*(i-1);
    alphan=m3*pi/W;
    alpha=sqrt(m3^2*pi^2/(W^2)+sqrt(Nx2(i,1)/D*m3^2*pi^2/(W^2)));
    beta=sqrt(-m3^2*pi^2/(W^2)+sqrt(Nx2(i,1)/D*m3^2*pi^2/(W^2)));

    A=[1 1 1 0; -alpha alpha 0 beta;...
       (alpha^2*exp(-alpha*L)-alphan^2*nu*exp(-alpha*L)) ...
       (alpha^2*exp(alpha*L)-alphan^2*nu*exp(alpha*L)) ...
       (-beta^2*cos(beta*L)-alphan^2*nu*cos(beta*L)) ...
       (-beta^2*sin(beta*L)-alphan^2*nu*sin(beta*L));...
       (-alpha^3*exp(-alpha*L)+alphan^2*(2-nu)*alpha*exp(-alpha*L)) ...
       (alpha^3*exp(alpha*L)-alphan^2*(2-nu)*alpha*exp(alpha*L)) ...
       (beta^3*sin(beta*L)+alphan^2*(2-nu)*beta*sin(beta*L)) ...
       (-beta^3*cos(beta*L)-alphan^2*(2-nu)*beta*cos(beta*L))];

    y2(i,1)=det(A);
end
```

Appendix G Continued

```
%Determinant Plot
plot(Nx,y,'*-',Nx1,y1,'d-',Nx2,y2,'p-')
xlabel('N_x','FontSize',14)
ylabel('Determinant','FontSize',14)
title('Plot of Determinant against N_x','FontSize',16)
legend('m = 1','m = 2','m = 3')
```

Determinant Behavior w.r.t. D

This m-file changes the value of the flexural rigidity and calculates the value of the determinant. This was used for initial guess purposes as well as a means to assess how changing the wavenumber changes the determinant behavior.

```
function AnalyticalBuckling3

%Change D and find the Determinant
clc
clear all
close all
m1=1; %Wavenumber
W=30; %Width of Gel
L=10; %Length of Gel
Nx=50;
D0=5; %Flexural Rigidity
for i=1:79
    D1(i,1)=D0+2.5*(i-1);

    alphas=m1*pi/W;
    nu=0.25; %Poisson Ratio
    alpha=sqrt(m1^2*pi^2/(W^2)+sqrt(Nx/D1(i,1)*m1^2*pi^2/(W^2)));
    beta=sqrt(-m1^2*pi^2/(W^2)+sqrt(Nx/D1(i,1)*m1^2*pi^2/(W^2)));

    A=[1 1 1 0; -alpha alpha 0 beta;...
        (alpha^2*exp(-alpha*L)-alphas^2*nu*exp(-alpha*L)) ...
        (alpha^2*exp(alpha*L)-alphas^2*nu*exp(alpha*L)) ...
        (-beta^2*cos(beta*L)-alphas^2*nu*cos(beta*L)) ...
        (-beta^2*sin(beta*L)-alphas^2*nu*sin(beta*L));...
        (-alpha^3*exp(-alpha*L)+alphas^2*(2-nu)*alpha*exp(-alpha*L)) ...
        (alpha^3*exp(alpha*L)-alphas^2*(2-nu)*alpha*exp(alpha*L)) ...
        (beta^3*sin(beta*L)+alphas^2*(2-nu)*beta*sin(beta*L)) ...
        (-beta^3*cos(beta*L)-alphas^2*(2-nu)*beta*cos(beta*L))];
    y1(i,1)=det(A);
end

D02=20;
m2=2;
```

Appendix G Continued

```
for i=1:73
    D2(i,1)=D02+2.5*(i-1);

    alphas=m2*pi/W;
    nu=0.25; %Poisson Ratio
    alpha=sqrt(m2^2*pi^2/(W^2)+sqrt(Nx/D2(i,1)*m2^2*pi^2/(W^2)));
    beta=sqrt(-m2^2*pi^2/(W^2)+sqrt(Nx/D2(i,1)*m2^2*pi^2/(W^2)));

    A=[1 1 1 0; -alpha alpha 0 beta;...
        (alpha^2*exp(-alpha*L)-alpham^2*nu*exp(-alpha*L)) ...
        (alpha^2*exp(alpha*L)-alpham^2*nu*exp(alpha*L)) ...
        (-beta^2*cos(beta*L)-alpham^2*nu*cos(beta*L)) ...
        (-beta^2*sin(beta*L)-alpham^2*nu*sin(beta*L));...
        (-alpha^3*exp(-alpha*L)+alpham^2*(2-nu)*alpha*exp(-alpha*L)) ...
        (alpha^3*exp(alpha*L)-alpham^2*(2-nu)*alpha*exp(alpha*L)) ...
        (beta^3*sin(beta*L)+alpham^2*(2-nu)*beta*sin(beta*L)) ...
        (-beta^3*cos(beta*L)-alpham^2*(2-nu)*beta*cos(beta*L))];

    y2(i,1)=det(A);
end

D03=40;
m3=3;
for i=1:65
    D3(i,1)=D03+2.5*(i-1);

    alphas=m3*pi/W;
    nu=0.25; %Poisson Ratio
    alpha=sqrt(m3^2*pi^2/(W^2)+sqrt(Nx/D3(i,1)*m3^2*pi^2/(W^2)));
    beta=sqrt(-m3^2*pi^2/(W^2)+sqrt(Nx/D3(i,1)*m3^2*pi^2/(W^2)));

    A=[1 1 1 0; -alpha alpha 0 beta;...
        (alpha^2*exp(-alpha*L)-alpham^2*nu*exp(-alpha*L)) ...
        (alpha^2*exp(alpha*L)-alpham^2*nu*exp(alpha*L)) ...
        (-beta^2*cos(beta*L)-alpham^2*nu*cos(beta*L)) ...
        (-beta^2*sin(beta*L)-alpham^2*nu*sin(beta*L));...
        (-alpha^3*exp(-alpha*L)+alpham^2*(2-nu)*alpha*exp(-alpha*L)) ...
        (alpha^3*exp(alpha*L)-alpham^2*(2-nu)*alpha*exp(alpha*L)) ...
        (beta^3*sin(beta*L)+alpham^2*(2-nu)*beta*sin(beta*L)) ...
        (-beta^3*cos(beta*L)-alpham^2*(2-nu)*beta*cos(beta*L))];

    y3(i,1)=det(A);
end
%Determinant Plot
plot(D1,y1,'*- ',D2,y2,'*- ',D3,y3,'*- ')
xlabel('(D) Flexural Rigidity','FontSize',14)
ylabel('Determinant','FontSize',14)
title('Plot of Determinant against D','FontSize',16)
legend('m = 1','m = 2','m = 3')
axis([0 200 -13 8])
```

Appendix G Continued

Determinant Behavior w.r.t. L/W

This m-file analyzes how the length to width ratio affects the determinant.

```
function AnalyticalBuckling6
clc
clear all
close all
m=1; %Wavenumber
W=30; %Width of Gel
L0=3;
Nx=20;
nu=0.25;
for i=1:38
    L(i,1)=L0+.5*(i-1);
    D=10; %Flexural Rigidity
    alphas=m*pi/W;
    alpha=sqrt(m^2*pi^2/(W^2)+sqrt(Nx/D*m^2*pi^2/(W^2)));
    beta=sqrt(-m^2*pi^2/(W^2)+sqrt(Nx/D*m^2*pi^2/(W^2)));

    A=[1 1 1 0; -alpha alpha 0 beta;...
        (alpha^2*exp(-alpha*L(i,1))-alphas^2*nu*exp(-alpha*L(i,1))) ...
        (alpha^2*exp(alpha*L(i,1))-alphas^2*nu*exp(alpha*L(i,1))) ...
        (-beta^2*cos(beta*L(i,1))-alphas^2*nu*cos(beta*L(i,1))) ...
        (-beta^2*sin(beta*L(i,1))-alphas^2*nu*sin(beta*L(i,1)));...
        (-alpha^3*exp(-alpha*L(i,1))+alphas^2*(2-nu)*alpha*exp(-
        alpha*L(i,1))) ...
        (alpha^3*exp(alpha*L(i,1))-alphas^2*(2-nu)*alpha*exp(alpha*L(i,1)))
        ...
        (beta^3*sin(beta*L(i,1))+alphas^2*(2-nu)*beta*sin(beta*L(i,1))) ...
        (-beta^3*cos(beta*L(i,1))-alphas^2*(2-nu)*beta*cos(beta*L(i,1)))]';

    y(i,1)=det(A);
end

%Determinant Plot
plot(L/W,y,'*-')
xlabel('L/W','FontSize',14)
ylabel('Determinant','FontSize',14)
title('Plot of Determinant against L/W','FontSize',16)
```

Appendix G Continued

Buckling Solution and Critical Buckling Magnitude

This m-file solves for the critical buckling solution at different values of the wavenumber.

This is done by solving the determinant expression for N_x and using the values that give rise to this solution to generate a plot of the critical buckling solution.

```
function AnalyticalBuckling2
clc
clear all
close all

%Generate Buckling Plot
m=5;
L=5;           %Length in y direction
W=30;         %Width in x direction
D=10;         %Flexural Rigidity
nu=0.25;      %Poisson Ratio
params.D=D;
params.l=L;
params.w=W;
params.nu=nu;

%Change the value of m and find smallest value of Nx for solution
for i=1:37
    mplot(i,1)=1+(i-1)*0.25;
    if i==1
        Nxsol=fzero(@(Nx)buckling(mplot(i,1),Nx,params),50);
    else
        Nxsol=fzero(@(Nx)buckling(mplot(i,1),Nx,params),Nxplot(i-1,1));
    end
    Nxplot(i,1)=Nxsol;
    if i>=2
        if Nxplot(i,1)< Nxplot(i-1,1)
            Nxmin=Nxplot(i,1);
            mmin=mplot(i,1);
        end
    end
end
disp(' ')
disp(['Critical Nx = ' num2str(Nxmin)])
disp(' ')
disp(['Critical Wavenumber = ' num2str(mmin)])
disp(' ')
%Stability Plot
plot(mplot,Nxplot,'*-')
xlabel('m','FontSize',14)
ylabel('N_x','FontSize',14)
```

Appendix G Continued

```
title('Plot of N_x against m','FontSize',16)

%Retrieve Constants from Constant Function
Cval=buckling2(mmin,Nxmin,params);
alpha=sqrt(mmin^2*pi^2/(W^2)+sqrt(Nxplot(17,1)/D*mmin^2*pi^2/(W^2)));
beta=sqrt(-mmin^2*pi^2/(W^2)+sqrt(Nxplot(17,1)/D*mmin^2*pi^2/(W^2)));
[X,Y]=meshgrid(0:0.75:W,0:0.75:L);
Z=sin(mmin.*pi.*X./W).*(Cval(1,1).*exp(-
alpha.*Y)+Cval(2,1).*exp(alpha.*Y)+Cval(3,1).*cos(beta.*Y)+Cval(4,1).*s
in(beta.*Y));

%Plot of Buckling Solution
figure;
surf(X,Y,Z)
colormap cool
colorbar
shading interp
axis equal
axis off
lightangle(30,30)
camorbit(-105,-3)
set(gcf,'Renderer','zbuffer')
set(findobj(gca,'type','surface'),...
'FaceLighting','phong',...
'AmbientStrength',0.3,'DiffuseStrength',.8,...
'SpecularStrength',.9,'SpecularExponent',40,...
'BackFaceLighting','unlit')

%Determinant Function
function f=buckling(m,Nx,params)
l=params.l; %Length of Gel
w=params.w; %Width of gel
alphan=m*pi/w;
nu=params.nu; %Poisson Ratio
Def=params.D; %Flexural Rigidity
alpha=sqrt(m^2*pi^2/(w^2)+sqrt(Nx/Def*m^2*pi^2/(w^2)));
beta=sqrt(-m^2*pi^2/(w^2)+sqrt(Nx/Def*m^2*pi^2/(w^2)));

A=1;
B=1;
C=1;
D=0;
E=-alpha;
F=alpha;
G=0;
H=beta;
I=(alpha^2*exp(-alpha*1)-alphan^2*nu*exp(-alpha*1));
J=(alpha^2*exp(alpha*1)-alphan^2*nu*exp(alpha*1));
K=(-beta^2*cos(beta*1)-alphan^2*nu*cos(beta*1));
L=(-beta^2*sin(beta*1)-alphan^2*nu*sin(beta*1));
M=(-alpha^3*exp(-alpha*1)+alphan^2*(2-nu)*alpha*exp(-alpha*1));
N=(alpha^3*exp(alpha*1)-alphan^2*(2-nu)*alpha*exp(alpha*1));
O=(beta^3*sin(beta*1)+alphan^2*(2-nu)*beta*sin(beta*1));
P=(-beta^3*cos(beta*1)-alphan^2*(2-nu)*beta*cos(beta*1));
```


Appendix G Continued

```
%Determinant of 4x4 Matrix
f=A*(F*(K*P-O*L)-G*(J*P-L*N)+H*(J*O-K*N))-...
    B*(E*(K*P-O*L)-G*(I*P-L*M)+H*(I*O-K*M))+...
    C*(E*(J*P-L*N)-F*(I*P-L*M)+H*(I*N-J*M))-...
    D*(E*(J*O-K*N)-F*(I*O-K*M)+G*(I*N-J*M));

%Constant Function
function C=buckling2(m,Nx,params)
L=params.l; %Length of Gel
W=params.w; %Width of gel
alphan=m*pi/W;
nu=params.nu; %Poisson Ratio
Def=params.D; %Flexural Rigidity
alpha=sqrt(m^2*pi^2/(W^2)+sqrt(Nx/Def*m^2*pi^2/(W^2)));
beta=sqrt(-m^2*pi^2/(W^2)+sqrt(Nx/Def*m^2*pi^2/(W^2)));

A=[1 1 1 0; -alpha alpha 0 beta;...
    (alpha^2*exp(-alpha*L)-alphan^2*nu*exp(-alpha*L)) ...
    (alpha^2*exp(alpha*L)-alphan^2*nu*exp(alpha*L)) ...
    (-beta^2*cos(beta*L)-alphan^2*nu*cos(beta*L)) ...
    (-beta^2*sin(beta*L)-alphan^2*nu*sin(beta*L));...
    (-alpha^3*exp(-alpha*L)+alphan^2*(2-nu)*alpha*exp(-alpha*L)) ...
    (alpha^3*exp(alpha*L)-alphan^2*(2-nu)*alpha*exp(alpha*L)) ...
    (beta^3*sin(beta*L)+alphan^2*(2-nu)*beta*sin(beta*L)) ...
    (-beta^3*cos(beta*L)-alphan^2*(2-nu)*beta*cos(beta*L))];
c=[0;0;0;0];
Amat=[A c];
%Making C4=1
Aref=rref(Amat);
C=[-Aref(1,4);-Aref(2,4);-Aref(3,4);1];
```

Change m and L/W to Determine Critical N_x

```
function AnalyticalBuckling9
clc
clear all
close all

%Generate Buckling Plot
L=7.5; %Length in y direction
W=30; %Width in x direction
D=10; %Flexural Rigidity
nu=0.25; %Poisson Ratio
params.D=D;
params.l=L;
params.w=W;
params.nu=nu;

%Change the value of m and find smallest value of  $N_x$  for solution
for i=1:37
```

Appendix G Continued

```
mplot(i,1)=1+(i-1)*0.25;
if i==1
    Nxsol=fzero(@(Nx)buckling(mplot(i,1),Nx,params),80);
else
    Nxsol=fzero(@(Nx)buckling(mplot(i,1),Nx,params),Nxplot(i-1,1));
end
Nxplot(i,1)=Nxsol;
end
L1=15;
params.l=L1;
for i=1:37
    mplot1(i,1)=1+(i-1)*0.25;
    if i==1
        Nxsol1=fzero(@(Nx)buckling(mplot1(i,1),Nx,params),50);
    else
        Nxsol1=fzero(@(Nx)buckling(mplot1(i,1),Nx,params),Nxplot1(i-1,1));
    end
    Nxplot1(i,1)=Nxsol1;
end
L2=30;
params.l=L2;
for i=1:37
    mplot2(i,1)=1+(i-1)*0.25;
    if i==1
        Nxsol2=fzero(@(Nx)buckling(mplot2(i,1),Nx,params),50);
    else
        Nxsol2=fzero(@(Nx)buckling(mplot2(i,1),Nx,params),Nxplot2(i-1,1));
    end
    Nxplot2(i,1)=Nxsol2;
end
%Stability Plot
plot(mplot,Nxplot,'*-',mplot1,Nxplot1,'d-',mplot2,Nxplot2,'p-')
xlabel('Wavenumber (m)','FontSize',14)
ylabel('N_x','FontSize',14)
title('Plot of N_x against m','FontSize',16)
legend('L/W = 0.25','L/W = 0.5','L/W = 1')

%Determinant Function
function f=buckling(m,Nx,params)
l=params.l; %Length of Gel
w=params.w; %Width of gel
alphan=m*pi/w;
nu=params.nu; %Poisson Ratio
Def=params.D; %Flexural Rigidity
alpha=sqrt(m^2*pi^2/(w^2)+sqrt(Nx/Def*m^2*pi^2/(w^2)));
beta=sqrt(-m^2*pi^2/(w^2)+sqrt(Nx/Def*m^2*pi^2/(w^2)));

A=1;
B=1;
C=1;
D=0;
E=-alpha;
```

Appendix G Continued

```
F=alpha;
G=0;
H=beta;
I=(alpha^2*exp(-alpha*1)-alphan^2*nu*exp(-alpha*1));
J=(alpha^2*exp(alpha*1)-alphan^2*nu*exp(alpha*1));
K=(-beta^2*cos(beta*1)-alphan^2*nu*cos(beta*1));
L=(-beta^2*sin(beta*1)-alphan^2*nu*sin(beta*1));
M=(-alpha^3*exp(-alpha*1)+alphan^2*(2-nu)*alpha*exp(-alpha*1));
N=(alpha^3*exp(alpha*1)-alphan^2*(2-nu)*alpha*exp(alpha*1));
O=(beta^3*sin(beta*1)+alphan^2*(2-nu)*beta*sin(beta*1));
P=(-beta^3*cos(beta*1)-alphan^2*(2-nu)*beta*cos(beta*1));
%Determinant of 4x4 Matrix
f=A*(F*(K*P-O*L)-G*(J*P-L*N)+H*(J*O-K*N))-...
    B*(E*(K*P-O*L)-G*(I*P-L*M)+H*(I*O-K*M))+...
    C*(E*(J*P-L*N)-F*(I*P-L*M)+H*(I*N-J*M))-...
    D*(E*(J*O-K*N)-F*(I*O-K*M)+G*(I*N-J*M));
```

Change m and D to Determine Critical N_x

```
function AnalyticalBuckling8
clc
clear all
close all

%Generate Buckling Plot
m=5;
L=5;           %Length in y direction
W=30;         %Width in x direction
D=10;         %Flexural Rigidity
nu=0.25;      %Poisson Ratio
params.D=D;
params.l=L;
params.w=W;
params.nu=nu;

%Change the value of m and find smallest value of  $N_x$  for solution
for i=1:37
    mplot(i,1)=1+(i-1)*0.25;
    if i==1
        Nxsol=fzero(@(Nx)buckling(mplot(i,1),Nx,params),50);
    else
        Nxsol=fzero(@(Nx)buckling(mplot(i,1),Nx,params),Nxplot(i-1,1));
    end
    Nxplot(i,1)=Nxsol;
end
D1=20;
params.D=D1;
for i=1:37
    mplot1(i,1)=1+(i-1)*0.25;
    if i==1
```

Appendix G Continued

```
Nxsol1=fzero(@(Nx)buckling(mplot1(i,1),Nx,params),50);
else
Nxsol1=fzero(@(Nx)buckling(mplot1(i,1),Nx,params),Nxplot1(i-
1,1));
end
Nxplot1(i,1)=Nxsol1;
end
D2=30;
params.D=D2;
for i=1:37
mplot2(i,1)=1+(i-1)*0.25;
if i==1
Nxsol2=fzero(@(Nx)buckling(mplot2(i,1),Nx,params),50);
else
Nxsol2=fzero(@(Nx)buckling(mplot2(i,1),Nx,params),Nxplot2(i-
1,1));
end
Nxplot2(i,1)=Nxsol2;
end
%Stability Plot
plot(mplot,Nxplot,'*- ',mplot1,Nxplot1,'d-',mplot2,Nxplot2,'p-')
xlabel('Wavenumber (m)','FontSize',14)
ylabel('N_x','FontSize',14)
title('Plot of N_x against m','FontSize',16)
legend('D = 10','D = 20','D = 30')

%Determinant Function
function f=buckling(m,Nx,params)
l=params.l; %Length of Gel
w=params.w; %Width of gel
alphan=m*pi/w;
nu=params.nu; %Poisson Ratio
Def=params.D; %Flexural Rigidity
alpha=sqrt(m^2*pi^2/(w^2)+sqrt(Nx/Def*m^2*pi^2/(w^2)));
beta=sqrt(-m^2*pi^2/(w^2)+sqrt(Nx/Def*m^2*pi^2/(w^2)));

A=1;
B=1;
C=1;
D=0;
E=-alpha;
F=alpha;
G=0;
H=beta;
I=(alpha^2*exp(-alpha*l)-alphan^2*nu*exp(-alpha*l));
J=(alpha^2*exp(alpha*l)-alphan^2*nu*exp(alpha*l));
K=(-beta^2*cos(beta*l)-alphan^2*nu*cos(beta*l));
L=(-beta^2*sin(beta*l)-alphan^2*nu*sin(beta*l));
M=(-alpha^3*exp(-alpha*l)+alphan^2*(2-nu)*alpha*exp(-alpha*l));
N=(alpha^3*exp(alpha*l)-alphan^2*(2-nu)*alpha*exp(alpha*l));
O=(beta^3*sin(beta*l)+alphan^2*(2-nu)*beta*sin(beta*l));
P=(-beta^3*cos(beta*l)-alphan^2*(2-nu)*beta*cos(beta*l));
%Determinant of 4x4 Matrix
f=A*(F*(K*P-O*L))-G*(J*P-L*N)+H*(J*O-K*N)-...
```

Appendix G Continued

```
B*(E*(K*P-O*L)-G*(I*P-L*M)+H*(I*O-K*M))+...  
C*(E*(J*P-L*N)-F*(I*P-L*M)+H*(I*N-J*M))-...  
D*(E*(J*O-K*N)-F*(I*O-K*M)+G*(I*N-J*M));
```

Change m and N_x to Determine Critical D

This m-file calculates the minimum critical value of flexural rigidity that will give rise to a buckling solution at different wavenumbers.

```
function AnalyticalBuckling4  
clc  
clear all  
close all  
  
L=5;           %Length in y direction  
W=30;         %Width in x direction  
Nx=30;  
nu=0.25;      %Poisson Ratio  
params.Nx=Nx;  
params.l=L;  
params.w=W;  
params.nu=nu;  
%Change the value of m and find smallest value of D for solution  
for i=1:37  
    mplot(i,1)=1+(i-1)*0.25;  
    if i==1  
        Dsol=fzero(@(Def)buckling(mplot(i,1),Def,params),20);  
    else  
        Dsol=fzero(@(Def)buckling(mplot(i,1),Def,params),Dplot(i-1,1));  
    end  
    Dplot(i,1)=Dsol;  
end  
Nx=40;  
params.Nx=Nx;  
for i=1:37  
    mplot1(i,1)=1+(i-1)*0.25;  
    if i==1  
        Dsol1=fzero(@(Def)buckling(mplot1(i,1),Def,params),20);  
    else  
        Dsol1=fzero(@(Def)buckling(mplot1(i,1),Def,params),Dplot1(i-  
1,1));  
    end  
    Dplot1(i,1)=Dsol1;  
end  
Nx=50;  
params.Nx=Nx;  
for i=1:37  
    mplot2(i,1)=1+(i-1)*0.25;
```

Appendix G Continued

```
if i==1
    Dsol2=fzero(@(Def)buckling(mplot2(i,1),Def,params),20);
else
    Dsol2=fzero(@(Def)buckling(mplot2(i,1),Def,params),Dplot2(i-
1,1));
end
Dplot2(i,1)=Dsol2;
end
```

```
%Stability Plot
plot(mplot,Dplot,'*- ',mplot1,Dplot1,'d-',mplot2,Dplot2,'p-')
xlabel('Wavenumber (m)','FontSize',14)
ylabel('Flexural Rigidity (D)','FontSize',14)
title('Plot of D against m','FontSize',16)
legend('N_x= 30','N_x= 40','N_x= 50')
```

```
%Determinant Function
function f=buckling(m,Def,params)
l=params.l; %Length of Gel
w=params.w; %Width of gel
alphan=m*pi/w;
nu=params.nu; %Poisson Ratio
Nx=params.Nx; %Flexural Rigidity
alpha=sqrt(m^2*pi^2/(w^2)+sqrt(Nx/Def*m^2*pi^2/(w^2)));
beta=sqrt(-m^2*pi^2/(w^2)+sqrt(Nx/Def*m^2*pi^2/(w^2)));

A=1;
B=1;
C=1;
D=0;
E=-alpha;
F=alpha;
G=0;
H=beta;
I=(alpha^2*exp(-alpha*l)-alphan^2*nu*exp(-alpha*l));
J=(alpha^2*exp(alpha*l)-alphan^2*nu*exp(alpha*l));
K=(-beta^2*cos(beta*l)-alphan^2*nu*cos(beta*l));
L=(-beta^2*sin(beta*l)-alphan^2*nu*sin(beta*l));
M=(-alpha^3*exp(-alpha*l)+alphan^2*(2-nu)*alpha*exp(-alpha*l));
N=(alpha^3*exp(alpha*l)-alphan^2*(2-nu)*alpha*exp(alpha*l));
O=(beta^3*sin(beta*l)+alphan^2*(2-nu)*beta*sin(beta*l));
P=(-beta^3*cos(beta*l)-alphan^2*(2-nu)*beta*cos(beta*l));
%Determinant of 4x4 Matrix
f=A*(F*(K*P-O*L)-G*(J*P-L*N)+H*(J*O-K*N))-...
B*(E*(K*P-O*L)-G*(I*P-L*M)+H*(I*O-K*M))+...
C*(E*(J*P-L*N)-F*(I*P-L*M)+H*(I*N-J*M))-...
D*(E*(J*O-K*N)-F*(I*O-K*M)+G*(I*N-J*M));
```

Appendix G Continued

Change m and N_x to Determine Critical L/W

This m-file calculates the minimum critical value of the length and width ratio that will give rise to a buckling solution at different wavenumbers.

```
function AnalyticalBuckling7
clc
clear all
close all

W=30;           %Width in x direction
Def=10;         %Flexural Rigidity
nu=0.25;       %Poisson Ratio
Nx=10;
params.Nx=Nx;
params.nu=nu;
params.w=W;
params.Def=Def;

%Change the value of m and find smallest value of L for solution
for i=1:33
    mplot(i,1)=1+(i-1)*0.25;
    if i==1
        Lsol=fzero(@(L)buckling(mplot(i,1),L,params),2);
    else
        Lsol=fzero(@(L)buckling(mplot(i,1),L,params),Lplot(i-1,1));
    end
    Lplot(i,1)=Lsol;
end
Nx=20;
params.Nx=Nx;
for i=1:48
    mplot1(i,1)=1+(i-1)*0.25;
    if i==1
        Lsol1=fzero(@(L)buckling(mplot1(i,1),L,params),2);
    else
        Lsol1=fzero(@(L)buckling(mplot1(i,1),L,params),Lplot1(i-1,1));
    end
    Lplot1(i,1)=Lsol1;
end
Nx=30;
params.Nx=Nx;
for i=1:30
    mplot2(i,1)=1+(i-1)*0.5;
    if i==1
        Lsol2=fzero(@(L)buckling(mplot2(i,1),L,params),2);
    else
        Lsol2=fzero(@(L)buckling(mplot2(i,1),L,params),Lplot2(i-1,1));
    end
end
```

Appendix G Continued

```
end
Lplot2(i,1)=Lsol2;
end
%Stability Plot
plot(mplot,Lplot/W,'*- ',mplot1,Lplot1/W,'d-',mplot2,Lplot2/W,'p-')
xlabel('m','FontSize',14)
ylabel('L / W','FontSize',14)
title('Plot of L / W against m','FontSize',16)
legend('N_x= 10','N_x= 20','N_x= 30')

%Determinant Function
function f=buckling(m,l,params)
nu=params.nu; %Poisson Ratio
w=params.w; %Width of gel
alphan=m*pi/w;
Def=params.Def; %Flexural Rigidity
Nx=params.Nx;
alpha=sqrt(m^2*pi^2/(w^2)+sqrt(Nx/Def*m^2*pi^2/(w^2)));
beta=sqrt(-m^2*pi^2/(w^2)+sqrt(Nx/Def*m^2*pi^2/(w^2)));

A=1;
B=1;
C=1;
D=0;
E=-alpha;
F=alpha;
G=0;
H=beta;
I=(alpha^2*exp(-alpha*1)-alphan^2*nu*exp(-alpha*1));
J=(alpha^2*exp(alpha*1)-alphan^2*nu*exp(alpha*1));
K=(-beta^2*cos(beta*1)-alphan^2*nu*cos(beta*1));
L=(-beta^2*sin(beta*1)-alphan^2*nu*sin(beta*1));
M=(-alpha^3*exp(-alpha*1)+alphan^2*(2-nu)*alpha*exp(-alpha*1));
N=(alpha^3*exp(alpha*1)-alphan^2*(2-nu)*alpha*exp(alpha*1));
O=(beta^3*sin(beta*1)+alphan^2*(2-nu)*beta*sin(beta*1));
P=(-beta^3*cos(beta*1)-alphan^2*(2-nu)*beta*cos(beta*1));
%Determinant of 4x4 Matrix
f=A*(F*(K*P-O*L)-G*(J*P-L*N)+H*(J*O-K*N))-...
B*(E*(K*P-O*L)-G*(I*P-L*M)+H*(I*O-K*M))+...
C*(E*(J*P-L*N)-F*(I*P-L*M)+H*(I*N-J*M))-...
D*(E*(J*O-K*N)-F*(I*O-K*M)+G*(I*N-J*M));
```

Strain Energy and Work Equation Generator

This m-file uses the Symbolic Math Toolbox in MATLAB to determine expressions for the work and strain energy of the hydrogel.

Appendix G Continued

```
clc
clear all

syms c1 c2 c3 c4 a b x y alphas alpha beta nu D Nx W L

%Solution to Biharmonic Equation
Z=sin(alphas*x)*(c1*exp(-
alpha*y)+c2*exp(alpha*y)+c3*cos(beta*y)+c4*sin(beta*y))

%Equation for Work Done on Hydrogel
Work=-Nx/2*int(int((diff(Z,x)^2),x,0,W),y,0,L);

%Simplified Work Expression
SimpleWork=simplify(Work)

%Differentials in Strain Energy Equation
diftwox=diff(diff(Z,x),x);
diftwoy=diff(diff(Z,y),y);
diftwoxy=diff(diff(Z,y),x);

%Strain Energy Equation
StrainEnergy=D/2*int(int((diftwox+diftwoy)^2-2*(1-
nu)*(diftwox*diftwoy...
-(diftwoxy)^2),y,0,L),x,0,W);

%Simplified Strain Energy Expression
SimpleStrainEnergy=simplify(StrainEnergy)
```

Strain Energy and Work Function

This m-file uses the determined expressions for work and strain energy to determine critical values of N_x at different wavenumbers. This is used as a means to verify that the results obtained are in agreement even though different methods were required to obtain them.

```
function StrainEnergy
clc
close all
clear all
W=30;
L=5;
D=10;
nu=0.25;
params.W=W;
params.L=L;
```

Appendix G Continued

```
params.nu=nu;
params.D=D;

N=18;
for i=1:N
    mval(i,1)=1+(i-1)*0.5;
    if i==1
        Nxsol(i,1)=fzero(@(Nx)buckling(mval(i,1),Nx,params),30);
    else
        Nxsol(i,1)=fzero(@(Nx)buckling(mval(i,1),Nx,params),Nxsol(i-
1,1));
    end
end

%Value of m to be analyzed
meval=1;

%Constants that cause buckling
C=buckling2(meval,Nxsol(meval+(meval-1),1),params);
params.c1=C(1);
params.c2=C(2);
params.c3=C(3);
params.c4=C(4);
for i=1:N+10
    Nxval(i,1)=0.5+(i-1)*1.25;
    work1(i,1)=workfun(meval,Nxval(i,1),params);
    strain1(i,1)=strainfun(meval,Nxval(i,1),params);
end
plot(Nxval,work1,'*- ',Nxval,strain1,'^-r')
xlabel('N_x','FontSize',14)
ylabel('Energy','FontSize',14)
title(['Plot of Energy against N_x when m = '
num2str(meval)],'FontSize',16)
legend('Work','Strain')

Nxcrit=fzero(@(Nx)(workfun(meval,Nx,params)-
(strainfun(meval,Nx,params))),30)

function f=workfun(m,Nx,params)
c1=params.c1;
c2=params.c2;
c3=params.c3;
c4=params.c4;
D=params.D;
W=params.W;
L=params.L;
alphan=m*pi/W;
alpha=sqrt(alpha^2+sqrt(Nx/D*alpha^2));
beta=sqrt(-alpha^2+sqrt(Nx/D*alpha^2));

f=1/64*Nx*alpha*(-4*c2^2*sin(2*alpha*W)*beta*alpha^2+8*c1*c4*...
exp(-alpha*L)*alpha*beta^2*sin(beta*L-2*alpha*W)-
8*c1^2*alpha*W*...
```


Appendix G Continued

```

sin(2*alphan*W)*L*alpha*beta^3+4*c4^2*sin(2*alphan*W)*L*alpha^3*...
beta+4*c4^2*sin(2*alphan*W)*L*alpha*beta^3+4*c3^2*sin(2*alphan*W)*L*...
alpha^3*beta+4*c3^2*sin(2*alphan*W)*L*alpha*beta^3-
8*c3*c4*alphan*W*...
cos(2*beta*L)*alpha^3-8*c3*c4*alphan*W*cos(2*beta*L)*alpha*beta^2-
8*...
c2*c4*exp(alpha*L)*alpha^2*beta*cos(beta*L+2*alphan*W)-8*c2*c4*...
exp(alpha*L)*alpha*beta^2*sin(beta*L+2*alphan*W)-8*c1*c4*exp(-
alpha*L)...
*alpha^2*beta*cos(beta*L-2*alphan*W)+8*c1*c3*exp(-
alpha*L)*alpha*...
beta^2*cos(beta*L-2*alphan*W)+8*c1*c3*exp(-alpha*L)*alpha^2*beta*...
*sin(beta*L-2*alphan*W)+8*c1*c4*exp(-alpha*L)*alpha^2*beta*...
cos(beta*L+2*alphan*W)-8*c1*c4*exp(-alpha*L)*alpha*beta^2*...
sin(beta*L+2*alphan*W)+8*c2*c3*exp(alpha*L)*alpha^2*beta*...
sin(beta*L+2*alphan*W)+8*c2*c3*exp(alpha*L)*alpha*beta^2*...
cos(beta*L-2*alphan*W)-
8*c2*c3*exp(alpha*L)*alpha^2*beta*sin(beta*L-...
2*alphan*W)+8*c2*c4*exp(alpha*L)*alpha^2*beta*cos(beta*L-
2*alphan*W)...
+8*c2*c4*exp(alpha*L)*alpha*beta^2*sin(beta*L-2*alphan*W)-
32*c1*c4*...
alphan*W*exp(-alpha*L)*alpha*beta^2*cos(beta*L)-
32*c1*c4*alphan*W*...
exp(-alpha*L)*alpha^2*beta*sin(beta*L)-32*c1*c3*alphan*W*...
exp(-alpha*L)*alpha^2*beta*cos(beta*L)+32*c1*c3*alphan*W*...
exp(-alpha*L)*alpha*beta^2*sin(beta*L)+32*c2*c3*alphan*W*...
exp(alpha*L)*alpha^2*beta*cos(beta*L)+32*c2*c3*alphan*W*exp(alpha*L)*...
.
alpha*beta^2*sin(beta*L)-
32*c2*c4*alphan*W*exp(alpha*L)*alpha*beta^2*...
cos(beta*L)+32*c2*c4*alphan*W*exp(alpha*L)*alpha^2*beta*sin(beta*L))...
/(alpha^2+beta^2)/alpha/beta;

```

```
function f=strainfun(m,Nx,params)
```

```

D=params.D;
c1=params.c1;
c2=params.c2;
c3=params.c3;
c4=params.c4;
W=params.W;
L=params.L;
nu=params.nu;
alphan=m*pi/W;
alpha=sqrt(alphan^2+sqrt(Nx/D*alphan^2));
beta=sqrt(-alphan^2+sqrt(Nx/D*alphan^2));

f=1/64*D*(32*alphan^5*W*c2*c4*beta^2*alpha+16*alpha*L*alphan^3*c4^2*...
beta^5*W+16*alpha*L*alphan^3*c3^2*beta^5*W+8*alpha*L*alphan^5*W*c4^2*...
.

```

Appendix G Continued

$$\begin{aligned}
 & \beta^3 + 8\alpha^L W c_4^2 \beta^7 \text{alphan} - 8\text{alphan}^2 \alpha^4 \beta^{\text{nu}} c_3^* \dots \\
 & c_2^* \exp(\alpha^L) \sin(\beta^L - \\
 & 2\text{alphan}^* W) + 8\text{alphan}^2 \alpha^4 \beta^{\text{nu}} c_3^* \dots \\
 & \quad c_1^* \exp(-\alpha^L) \sin(\beta^L - 2\text{alphan}^* W) - 8\text{alphan}^4 c_1^* c_4^* \dots \\
 & \quad \exp(-\alpha^L) \alpha \beta^2 \sin(\beta^L - 2\text{alphan}^* W) - 16\text{alphan}^2 \dots \\
 & \quad \alpha^2 \beta^3 c_2^* c_4^* \exp(\alpha^L) \cos(\beta^L + 2\text{alphan}^* W) - 8c_1^2 \dots \\
 & \quad \alpha^6 \exp(-\alpha^L)^2 \beta^W \text{alphan} - 32c_2^* \alpha^3 c_3^* \beta^4 \dots \\
 & \quad \exp(\alpha^L) \sin(\beta^L) W \text{alphan} + 8\beta^4 W c_3^* c_4^* \text{alphan} \alpha^3 - \\
 & 16^* \dots \\
 & \quad \beta^2 \text{alphan}^3 W c_3^* c_4^* \alpha^3 + 32\beta^2 \text{alphan}^3 \text{nu} c_3^* c_4^* W \dots \\
 & \alpha^3 + 8\text{alphan}^5 W c_3^* c_4^* \alpha^3 + 8\alpha^L \text{alphan}^5 W c_3^2 \beta^3 + \dots \\
 & 32\alpha^L \text{alphan}^5 c_1^* c_2^* W \beta^3 + 32\beta^4 \text{alphan}^3 \text{nu} c_3^* c_4^* W \dots \\
 & \quad \alpha + 8\beta^6 W c_3^* c_4^* \text{alphan} \alpha - 8\alpha^6 c_2^2 W \beta \text{alphan} - \\
 & 16^* \dots \\
 & \text{alphan}^4 \sin(2\text{alphan}^* W) \beta^2 c_1^* c_4^* \alpha + 8c_1^2 \alpha^6 W \beta \dots \\
 & \quad \text{alphan} - \\
 & 4\beta^4 c_4^2 \sin(2\beta^L) \alpha^3 W \text{alphan} + 32c_2^* \alpha^3 c_4^* \dots \\
 & \quad \beta^4 \exp(\alpha^L) \cos(\beta^L) W \text{alphan} - \\
 & 24\text{alphan}^2 \alpha^2 \beta^3 \dots \\
 & \quad \text{nu} c_1^* c_3^* \exp(-\alpha^L) \sin(\beta^L - 2\text{alphan}^* W) - 32\text{alphan}^5 c_1^* c_4^* \dots \\
 & \quad \exp(-\alpha^L) \alpha^2 \beta \sin(\beta^L) W - 32\text{alphan}^5 c_1^* c_4^* \dots \\
 & \quad \exp(-\alpha^L) \alpha \beta^2 \cos(\beta^L) W - 8\text{alphan}^4 c_2^* c_4^* \dots \\
 & \quad \exp(\alpha^L) \alpha^2 \beta \cos(\beta^L - \\
 & 2\text{alphan}^* W) + 8\alpha^L W c_3^2 \dots \\
 & \quad \beta^7 \text{alphan} - \\
 & 32\text{alphan}^3 c_1^* \alpha^4 \text{nu} W c_3^* \beta + 16\sin(2\text{alphan}^* W) \dots \\
 & \quad c_1^* \alpha^4 c_3^* \beta^3 + 16\text{alphan}^3 W c_1^2 \alpha^2 \beta^3 - \\
 & 8\text{alphan}^5 \dots \\
 & \quad c_2^2 W \beta \alpha^2 - 4\text{alphan}^4 c_2^2 \exp(\alpha^L)^2 \beta \alpha^2 \dots \\
 & \quad \sin(2\text{alphan}^* W) + 8\text{alphan}^2 \alpha \beta^4 \text{nu} c_1^* c_3^* \exp(-\alpha^L) \dots \\
 & \quad \cos(\beta^L + 2\text{alphan}^* W) - \\
 & 8\text{alphan}^2 \alpha^4 \beta \text{nu} c_4^* c_2^* \exp(\alpha^L) \dots \\
 & \quad \cos(\beta^L + 2\text{alphan}^* W) - 32\text{alphan}^3 \alpha^3 \beta^2 \text{nu} c_1^* c_4^* \dots \\
 & \quad \exp(- \\
 & \alpha^L) \cos(\beta^L) W + 8\text{alphan}^5 c_2^2 \exp(\alpha^L)^2 \beta^3 W + \dots \\
 & \quad 32\text{alphan}^5 c_1^* \alpha^2 W c_3^* \beta + 4\alpha^6 \sin(2\text{alphan}^* W) c_2^2 \dots \\
 & \beta + 4\text{alphan}^4 \sin(2\text{alphan}^* W) c_2^2 \beta^3 + 32\text{alphan}^3 \text{nu} c_2^* c_3^* \dots \\
 & \alpha^4 W \beta + 32\alpha^4 \text{alphan}^3 c_2^2 W \text{nu} \beta + 32\alpha^2 \text{alphan}^4 \dots \\
 & \quad 3c_2^2 W \text{nu} \beta^3 - 4c_1^2 \alpha^4 \sin(2\text{alphan}^* W) \beta^3 - 4c_4^2 \dots \\
 & \text{alphan}^5 \sin(2\beta^L) \alpha \beta^2 W + 32\text{alphan}^3 \alpha^4 \beta \text{nu} \dots \\
 & \quad c_3^* c_1^* \exp(-\alpha^L) \cos(\beta^L) W - \\
 & 32\text{alphan}^3 \alpha^4 \beta \text{nu} c_3^* c_2^* \dots \\
 & \quad \exp(\alpha^L) \cos(\beta^L) W - \\
 & 8\text{alphan}^4 c_2^* c_4^* \exp(\alpha^L) \alpha \beta \dots \\
 & \quad 2\sin(\beta^L - 2\text{alphan}^* W) + 16\text{alphan}^2 \alpha^3 \beta^2 c_3^* c_2^* \dots \\
 & \exp(\alpha^L) \cos(\beta^L + 2\text{alphan}^* W) + 8\text{alphan}^2 \alpha^4 \beta \text{nu} c_4^* \dots \\
 & \quad c_1^* \exp(-\alpha^L) \cos(\beta^L + 2\text{alphan}^* W) - \\
 & 16\text{alphan}^3 \alpha \beta^4 \text{nu} \dots
 \end{aligned}$$

Appendix G Continued

$$\begin{aligned}
 & c4^2 \sin(2 \beta L) W + 8 \alpha^5 W^3 c3^4 \alpha \beta^2 + 16 \sin(2 \alpha W) \dots \\
 & \dots \\
 & W) \beta^4 c1 \alpha^3 c4 - \\
 & 16 \sin(2 \alpha W) c2 \alpha^4 c3 \beta^3 + 16 \dots \\
 & \alpha^2 \sin(2 \alpha W) c1 \alpha^4 \nu c3 \beta - \\
 & 32 c1 \alpha^4 \beta^3 \dots \\
 & W c3 \alpha + 32 \alpha^3 \nu c2 c3 \beta^3 W \alpha^2 + 48 \alpha^2 \dots \\
 & \sin(2 \alpha W) \nu c2 \alpha^3 c4 \beta^2 - \\
 & 32 \alpha^2 \sin(2 \alpha W) \dots \\
 & c2 \alpha^3 c4 \beta^2 - 48 \alpha^2 \sin(2 \alpha W) \beta^3 c1 \nu \dots \\
 & \alpha^2 c3 - 8 \alpha^5 c2^2 W \beta^3 - \\
 & 64 \alpha^3 c2 \alpha^2 c3 \beta^3 \dots \\
 & W + 32 \alpha^3 \nu c2 \alpha^3 c4 \beta^2 W - 8 \alpha^2 \alpha^2 \dots \\
 & \sin(2 \alpha W) c2^2 \beta^3 + 32 \alpha^2 \sin(2 \alpha W) \beta^3 c1 \dots \\
 & \alpha^2 c3 + 48 \alpha^2 \sin(2 \alpha W) \beta^2 c1 \nu \alpha^3 c4 + 4 \dots \\
 & c3^2 \alpha^5 \sin(2 \beta L) \alpha^3 W + 24 \alpha^2 \alpha^3 \beta^2 \nu \dots \\
 & c1 c3 \exp(-\alpha L) \cos(\beta L - 2 \alpha W) + 4 c3^2 \alpha^5 \dots \\
 & \sin(2 \beta L) \alpha \beta^2 W - \\
 & 8 \alpha^5 c3 c4 \cos(2 \beta L) \alpha \dots \\
 & \beta^2 W + 8 \alpha^4 c2 c4 \exp(\alpha L) \alpha^2 \beta \dots \\
 & \cos(\beta L + 2 \alpha W) + 16 \alpha^2 \alpha^2 \beta^3 c1 c4 \dots \\
 & \exp(-\alpha L) \cos(\beta L + 2 \alpha W) + c3^2 \alpha^4 \alpha \beta^2 \dots \\
 & \cos(2 \beta L + 2 \alpha W) + 32 c1 \alpha^3 c4 \beta^4 \exp(-\alpha L) \dots \\
 & \cos(\beta L) W \alpha - 8 \alpha^4 c2^2 W \beta^3 \alpha - 4 \alpha^4 L \dots \\
 & \alpha^4 \sin(2 \alpha W) c3^2 \beta^3 + 8 c2 \alpha^3 c3 \beta^4 \dots \\
 & \exp(\alpha L) \cos(\beta L - 2 \alpha W) - 2 c3 \beta^4 c4 \alpha^3 \dots \\
 & \sin(2 \beta L - 2 \alpha W) - 8 c2 \alpha^3 \alpha^3 c4 \beta^4 \exp(\alpha L) \dots \\
 & \sin(\beta L + 2 \alpha W) - 2 c3 \beta^6 c4 \alpha \sin(2 \beta L - 2 \dots \\
 & \alpha W) + c3^2 \alpha^4 \alpha^3 \cos(2 \beta L + 2 \alpha W) - \\
 & 32 c1^2 \dots \\
 & \alpha^4 \alpha^3 W \nu \beta - \\
 & 32 \alpha^3 \nu c1^2 \alpha^2 W \beta^3 + 4 \dots \\
 & \alpha^4 \sin(2 \alpha W) c2^2 \beta^3 - \\
 & 16 \alpha^2 \alpha^2 \beta^3 c1 \dots \\
 & c3 \exp(-\alpha L) \sin(\beta L + 2 \alpha W) + 4 \alpha^2 c3 c4 \beta^2 \dots \\
 & \alpha^3 \sin(2 \beta L + 2 \alpha W) + 2 \beta^2 c4^2 \alpha^2 \alpha^3 \dots \\
 & \cos(2 \beta L - 2 \alpha W) - 2 \alpha^2 \alpha \beta^4 c3^2 \dots \\
 & \cos(2 \beta L - 2 \alpha W) + 24 \alpha^2 \alpha^2 \beta^3 \nu c2 c3 \dots \\
 & \exp(\alpha L) \sin(\beta L - \\
 & 2 \alpha W) + 64 \alpha^3 \alpha^2 \beta^3 c2 \dots \\
 & c4 \exp(\alpha L) \sin(\beta L) W - 2 \alpha^2 \alpha^3 \beta^2 c3^2 \dots \\
 & \cos(2 \beta L - 2 \alpha W) - 4 L \sin(2 \alpha W) \beta^7 c3^2 \dots \\
 & \alpha - \beta^4 c3^2 \alpha^3 \cos(2 \beta L - 2 \alpha W) - \\
 & 32 \alpha^3 \alpha \dots \\
 & \beta^4 \nu c1 c4 \exp(-\alpha L) \cos(\beta L) W + 24 \alpha^2 \alpha^2 \dots \\
 & \beta^3 \nu c2 c4 \exp(\alpha L) \cos(\beta L + 2 \alpha W) - 32 \alpha^3 \dots \\
 & \alpha^2 \beta^3 \nu c2^2 \exp(\alpha L)^2 W - \\
 & 16 \alpha^3 \alpha^2 \beta^3 \dots \\
 & c1^2 \exp(-\alpha L)^2 W + 24 \alpha^2 \alpha^3 \beta^2 \nu c3 c2 \dots \\
 & \exp(\alpha L) \cos(\beta L - 2 \alpha W) - \\
 & 8 \alpha^5 c3 c4 \cos(2 \beta L) \dots
 \end{aligned}$$

Appendix G Continued

$$\begin{aligned}
 & \alpha^3 W + 8 \text{alphan}^5 W c1^2 \beta^3 - 32 \text{alphan}^5 c1 c3 \exp(-\alpha L) \dots \\
 & \alpha^2 \beta \cos(\beta L) W + 32 \text{alphan}^5 c1 c3 \exp(-\alpha L) \alpha \dots \\
 & \beta^2 \sin(\beta L) W + 4 \text{alphan}^2 c3 c4 \beta^4 \alpha \dots \\
 & \sin(2\beta L + 2\text{alphan} W) - 8 \text{alphan}^2 \alpha \beta^4 \nu c2 c4 \dots \\
 & \exp(\alpha L) \sin(\beta L - 2\text{alphan} W) - 8 \text{alphan}^4 c1 c3 \exp(-\alpha L) \dots \\
 & \alpha^2 \beta \sin(\beta L - 2\text{alphan} W) - 24 \text{alphan}^2 \alpha^2 \beta^3 \nu \dots \\
 & c2 c4 \exp(\alpha L) \cos(\beta L - 2\text{alphan} W) + 8 \text{alphan}^4 c2 c3 \dots \\
 & \exp(\alpha L) \alpha \beta^2 \cos(\beta L + 2\text{alphan} W) + 8 \text{alphan}^2 c1^2 \dots \\
 & \alpha^2 \sin(2\text{alphan} W) \beta^3 - 32 \text{alphan}^2 \sin(2\text{alphan} W) c2 \dots \\
 & \alpha^2 c3 \beta^3 - 32 \text{alphan}^3 \beta^3 c1 \nu \alpha^2 c3 W - 32 \beta^4 \dots \\
 & c1 W \alpha^3 c4 \text{alphan} - 64 \text{alphan}^3 c1 \alpha^3 \beta^2 W c4 + 32 \alpha^7 L c1 c2 W \text{alphan} \beta - \\
 & 64 \alpha^5 L \text{alphan}^3 c1 c2 W \beta + 16 \text{alphan}^4 \dots \\
 & 4 \sin(2\text{alphan} W) c2 c3 \alpha^2 \beta + 32 \text{alphan}^5 \beta^2 c1 W c4 \dots \\
 & \alpha + 32 \alpha^5 L c1 c2 W \text{alphan} \beta^3 - 16 \beta^4 \text{alphan}^3 W c3 \dots \\
 & c4 \alpha + 32 \beta^4 \text{alphan}^3 c1 \nu c4 W \alpha - 24 \text{alphan}^2 \alpha^3 \dots \\
 & \beta^2 \nu c1 c3 \exp(-\alpha L) \cos(\beta L + 2\text{alphan} W) + 4 \text{alphan}^4 \dots \\
 & c1^2 \exp(-\alpha L)^2 \beta \alpha^2 \sin(2\text{alphan} W) - 64 \text{alphan}^3 c1 c4 \dots \\
 & \beta^3 \exp(-\alpha L) \alpha^2 \sin(\beta L) W + 8 c2 \alpha^3 c4 \beta^4 \dots \\
 & \exp(\alpha L) \sin(\beta L - 2\text{alphan} W) - 2 \text{alphan}^4 c3 c4 \alpha^3 \dots \\
 & \sin(2\beta L - 2\text{alphan} W) + 8 \text{alphan}^2 \alpha \beta^4 \nu c1 c4 \dots \\
 & \exp(-\alpha L) \sin(\beta L + 2\text{alphan} W) + 8 \text{alphan}^5 W c1^2 \beta \alpha^2 - \dots \\
 & 4 \text{alphan}^4 c1^2 \sin(2\text{alphan} W) \beta^3 + 32 \text{alphan}^3 \nu c2 c4 \beta^4 \dots \\
 & W \alpha + 48 \text{alphan}^2 \sin(2\text{alphan} W) \nu c3 c2 \beta^3 \alpha^2 + 4 \text{alphan}^4 \dots \\
 & 4 \sin(2\text{alphan} W) c2^2 \beta \alpha^2 - 4 c2^2 \alpha^4 \exp(\alpha L)^2 \dots \\
 & \beta^3 \sin(2\text{alphan} W) + 4 c1^2 \alpha^6 \exp(-\alpha L)^2 \beta \dots \\
 & \sin(2\text{alphan} W) - 32 W c2 \alpha^3 c4 \beta^4 \text{alphan} - 32 \text{alphan}^2 \dots \\
 & \sin(2\text{alphan} W) c1 \alpha^3 c4 \beta^2 - 8 \alpha^4 \text{alphan}^2 \dots \\
 & \sin(2\text{alphan} W) c2^2 \beta - 16 \text{alphan}^2 \sin(2\text{alphan} W) \nu c3 c2 \dots \\
 & \alpha^4 \beta - 8 \text{alphan}^2 \alpha^4 \beta \nu c4 c1 \exp(-\alpha L) \dots \\
 & \cos(\beta L - 2\text{alphan} W) - c4^2 \text{alphan}^4 \alpha \beta^2 \dots \\
 & \cos(2\beta L + 2\text{alphan} W) + 32 \text{alphan}^3 \alpha^4 \beta \nu c4 c1 \dots \\
 & \exp(-\alpha L) \sin(\beta L) W - c3^2 \text{alphan}^4 \alpha \beta^2 \dots \\
 & \cos(2\beta L - 2\text{alphan} W) - 32 \text{alphan}^3 \alpha \beta^4 \nu c3 c4 \dots \\
 & \cos(2\beta L) W + 64 \text{alphan}^3 \alpha^3 \beta^2 c1 c4 \exp(-\alpha L) \dots \\
 & \cos(\beta L) W - 8 \text{alphan}^2 \alpha \beta^4 \nu c1 c3 \exp(-\alpha L) \dots \\
 & \cos(\beta L - 2\text{alphan} W) + 8 c1^2 \alpha^4 \text{alphan}^2 \sin(2\text{alphan} W) \beta - \dots
 \end{aligned}$$

Appendix G Continued

$$\begin{aligned}
 & 2^*\beta^4*c^4^2*\alpha^2*\alpha*\cos(2*\beta*L+2*\alpha*W)+24*\alpha^2*\dots \\
 & \quad \alpha^2*\beta^3*\nu*c^1*c^4*\exp(-\alpha*L)*\cos(\beta*L-2*\alpha*W)+64*\dots \\
 & \alpha^3*\alpha^3*\beta^2*c^4*c^2*\exp(\alpha*L)*\cos(\beta*L)*W+16*\alpha^2*\dots \\
 & \quad \alpha^3*\beta^2*c^3*c^1*\exp(-\alpha*L)*\cos(\beta*L+2*\alpha*W)- \\
 & 8*c^2*\alpha^*\dots \\
 & \quad 4*c^4*\beta^3*\exp(\alpha*L)*\cos(\beta*L+2*\alpha*W)- \\
 & 24*\alpha^2*\alpha^*\dots \\
 & \quad 3*\beta^2*\nu*c^3*c^2*\exp(\alpha*L)*\cos(\beta*L+2*\alpha*W)- \\
 & 24*\alpha^2*\dots \\
 & \quad \alpha^3*\beta^2*\nu*c^1*c^4*\exp(- \\
 & \alpha*L)*\sin(\beta*L+2*\alpha*W)+\beta^4*\dots \\
 & \quad c^3^2*\alpha^3*\cos(2*\beta*L+2*\alpha*W)+8*c^1^2*\alpha^4*W*\beta^3*\dots \\
 & \quad \alpha^3-16*\alpha^2*\alpha^2*\beta^3*c^1*c^4*\exp(-\alpha*L)*\dots \\
 & \quad \cos(\beta*L-2*\alpha*W)+16*\alpha^3*c^2^2*\alpha^4*\exp(\alpha*L)^2*\dots \\
 & \quad \beta*W-32*\alpha^3*\alpha^3*\beta^2*\nu*c^3*c^4*\cos(2*\beta*L)*W- \\
 & 2*\beta^2*\dots \\
 & \quad c^4^2*\alpha^2*\alpha^3*\cos(2*\beta*L+2*\alpha*W)+2*\alpha^4*c^3*c^4*\dots \\
 & \alpha^3*\sin(2*\beta*L+2*\alpha*W)+16*\alpha^3*\alpha^3*\beta^2*c^3*c^4*\dots \\
 & \quad \cos(2*\beta*L)*W+64*\alpha^3*\alpha^2*\beta^3*c^2*c^3*\exp(\alpha*L)*\dots \\
 & \quad \cos(\beta*L)*W-16*\alpha^4*\sin(2*\alpha*W)*c^2*c^4*\beta^2*\alpha+32*\dots \\
 & \beta^2*\alpha^3*c^1*\nu*c^4*\alpha^3*W+16*\alpha^3*L*\alpha^3*c^3^2*\beta^3*\dots \\
 & \quad W+64*\alpha^3*\beta^3*c^1*\alpha^2*c^3*W- \\
 & 16*\alpha^2*\sin(2*\alpha*W)*\nu*\dots \\
 & \quad c^2*c^4*\beta^4*\alpha+16*\alpha^3*L*\alpha^3*c^4^2*\beta^3*W-16*L*\dots \\
 & \quad \sin(2*\alpha*W)*\alpha^7*c^2*c^1*\beta- \\
 & 16*L*\sin(2*\alpha*W)*\alpha^5*c^2*\dots \\
 & \quad c^1*\beta^3+64*\alpha^5*L*\alpha^2*\sin(2*\alpha*W)*\nu*c^2*c^1*\beta- \\
 & 32*\dots \\
 & \quad \alpha^5*L*\alpha^2*\sin(2*\alpha*W)*c^2*c^1*\beta- \\
 & 8*c^1*\alpha^3*c^3*\beta^4*\dots \\
 & \quad \exp(-\alpha*L)*\cos(\beta*L+2*\alpha*W)-8*c^2*\alpha^4*c^3*\beta^3*\dots \\
 & \quad \exp(\alpha*L)*\sin(\beta*L-2*\alpha*W)-4*c^3*\beta^4*c^4*\alpha^3*\dots \\
 & \quad \sin(2*\alpha*W)- \\
 & 24*\alpha^2*\alpha^3*\beta^2*\nu*c^4*c^2*\exp(\alpha*L)*\dots \\
 & \sin(\beta*L+2*\alpha*W)+16*\alpha^2*\alpha^3*\beta^2*c^4*c^2*\exp(\alpha*L)*\dots \\
 & \quad \sin(\beta*L+2*\alpha*W)+24*\alpha^2*\alpha^3*\beta^2*\nu*c^4*c^2*\dots \\
 & \quad \exp(\alpha*L)*\sin(\beta*L-2*\alpha*W)-8*\alpha^5*c^1^2*\dots \\
 & \quad \exp(-\alpha*L)^2*\beta*\alpha^2*W+4*\alpha^4*c^1^2*\exp(-\alpha*L)^2*\dots \\
 & \quad \beta^3*\sin(2*\alpha*W)-8*\alpha^2*\alpha^4*\beta*\nu*c^3*c^1*\dots \\
 & \quad \exp(-\alpha*L)*\sin(\beta*L+2*\alpha*W)-64*\alpha^3*c^2*\alpha^3*c^4*\dots \\
 & \beta^2*W+8*\alpha^3*L*\alpha^5*W*c^4^2*\beta+8*\alpha^3*L*W*c^4^2*\beta^5*\dots \\
 & \alpha^3+32*\alpha^5*c^2*c^3*\exp(\alpha*L)*\alpha^2*\beta*\cos(\beta*L)*W+24*\dots \\
 & \quad \alpha^2*\alpha^2*\beta^3*\nu*c^1*c^3*\exp(-\alpha*L)*\dots \\
 & \quad \sin(\beta*L+2*\alpha*W)+16*\alpha^2*\alpha^3*\beta^2*c^1*c^4*\exp(- \\
 & \alpha*L)*\dots
 \end{aligned}$$

Appendix G Continued

$$\begin{aligned} & \sin(\beta * L + 2 * \alpha * W) - \\ & 16 * \alpha * \alpha^2 * \beta^3 * c^2 * c^3 * \exp(\alpha * L) * \dots \\ & \sin(\beta * L - 2 * \alpha * W) - 16 * \alpha * \alpha^2 * \beta^3 * c^2 * c^3 * \dots \\ & \exp(\alpha * L) * \cos(\beta * L - \\ & 2 * \alpha * W) + 8 * \alpha * \alpha^4 * c^2 * c^4 * \exp(\alpha * L) * \dots \\ & \alpha * \beta^2 * \sin(\beta * L + 2 * \alpha * W) + 8 * \alpha * \alpha^2 * c^2 * \alpha^4 * \dots \\ & \exp(\alpha * L)^2 * \beta * \sin(2 * \alpha * W) + 8 * c^1 * \alpha^4 * c^4 * \beta^3 * \dots \\ & \exp(-\alpha * L) * \cos(\beta * L + 2 * \alpha * W) + 8 * \alpha * L * \alpha^2 * \dots \\ & \sin(2 * \alpha * W) * \beta^5 * c^3^2 + 64 * \alpha^3 * L * \alpha^2 * \sin(2 * \alpha * W) * \nu * \dots \\ & \dots \\ & c^2 * c^1 * \beta^3 + 8 * \alpha^5 * c^2^2 * \exp(\alpha * L)^2 * \beta * \alpha^2 * W - \\ & 4 * \alpha^2 * \dots \\ & c^3 * c^4 * \beta^2 * \alpha^3 * \sin(2 * \beta * L - 2 * \alpha * W) + \beta^6 * c^4^2 * \alpha * \dots \\ & \cos(2 * \beta * L - 2 * \alpha * W) - 8 * c^1 * \alpha^4 * c^4 * \beta^3 * \exp(-\alpha * L) * \dots \\ & \cos(\beta * L - 2 * \alpha * W) + 32 * \alpha^3 * \alpha * \beta^4 * \nu * c^1 * c^3 * \dots \\ & \exp(- \\ & \alpha * L) * \sin(\beta * L) * W + 32 * \alpha^5 * c^2 * c^4 * \exp(\alpha * L) * \alpha^2 * \dots \\ & \beta * \sin(\beta * L) * W + 32 * \alpha^3 * \alpha^2 * \beta^3 * \nu * c^1 * c^3 * \dots \\ & \exp(-\alpha * L) * \cos(\beta * L) * W + 16 * \alpha^3 * \alpha^3 * \beta^2 * \nu * c^3^2 * \dots \\ & \sin(2 * \beta * L) * W - 32 * \alpha^5 * c^2 * c^4 * \exp(\alpha * L) * \alpha * \beta^2 * \dots \\ & \cos(\beta * L) * W - 8 * \alpha^2 * \alpha^2 * \beta^3 * c^1^2 * \exp(-\alpha * L)^2 * \dots \\ & \sin(2 * \alpha * W) + 16 * \sin(2 * \alpha * W) * c^2 * \alpha^3 * c^4 * \beta^4 - \\ & 32 * \alpha^3 * \dots \\ & \alpha^4 * \beta * \nu * c^2^2 * \exp(\alpha * L)^2 * W + 32 * \alpha^3 * \alpha^3 * \beta^2 * \nu * \dots \\ & c^3 * c^2 * \exp(\alpha * L) * \sin(\beta * L) * W - \\ & 64 * \alpha^3 * \alpha^2 * \beta^3 * c^1 * c^3 * \dots \\ & \exp(-\alpha * L) * \cos(\beta * L) * W - 8 * \alpha^5 * \alpha^2 * \alpha * \beta^4 * \nu * c^2 * c^3 * \dots \\ & \exp(\alpha * L) * \cos(\beta * L - 2 * \alpha * W) + c^4^2 * \alpha^4 * \alpha * \beta^2 * \dots \\ & \cos(2 * \beta * L - 2 * \alpha * W) + 8 * \alpha^2 * \alpha^4 * \nu * c^2 * c^3 * \dots \\ & \exp(\alpha * L) * \cos(\beta * L + 2 * \alpha * W) - \\ & 8 * \alpha^2 * c^3 * c^4 * \beta^2 * \alpha^3 * \dots \\ & \sin(2 * \alpha * W) - \\ & 4 * c^1^2 * \alpha^6 * \sin(2 * \alpha * W) * \beta + 32 * \alpha^5 * c^2 * \dots \\ & c^3 * \exp(\alpha * L) * \alpha * \beta^2 * \sin(\beta * L) * W - \\ & 8 * \alpha^2 * c^3 * c^4 * \beta^4 * \dots \\ & \alpha * \sin(2 * \alpha * W) + 8 * \alpha^2 * c^2^2 * \alpha^2 * \exp(\alpha * L)^2 * \beta^3 * \dots \\ & \sin(2 * \alpha * W) + 8 * \alpha^2 * \alpha^4 * \beta * \nu * c^3 * c^2 * \exp(\alpha * L) * \dots \\ & \sin(\beta * L + 2 * \alpha * W) + 8 * \alpha^4 * c^1 * c^4 * \exp(- \\ & \alpha * L) * \alpha * \beta^2 * \dots \\ & \sin(\beta * L + 2 * \alpha * W) + 8 * c^2^2 * \alpha^6 * \exp(\alpha * L)^2 * \beta * W * \alpha + 32 * \dots \\ & \dots \\ & \alpha^3 * L * \alpha^5 * c^1 * c^2 * W * \beta - 4 * \alpha^2 * c^3 * c^4 * \beta^4 * \alpha * \dots \\ & \sin(2 * \beta * L - 2 * \alpha * W) - 8 * \alpha^4 * c^1 * c^3 * \exp(-\alpha * L) * \alpha * \dots \\ & \beta^2 * \cos(\beta * L - 2 * \alpha * W) - 8 * \alpha^2 * \alpha^2 * \beta^4 * \nu * c^1 * c^4 * \dots \\ & \exp(-\alpha * L) * \sin(\beta * L - 2 * \alpha * W) - 8 * \alpha^5 * c^1^2 * \exp(- \\ & \alpha * L)^2 * \dots \\ & \beta^3 * W - 16 * \alpha^3 * \alpha^4 * \beta * c^1^2 * \exp(-\alpha * L)^2 * W - \\ & 32 * \alpha^3 * \dots \\ & \alpha^2 * \beta^3 * \nu * c^2 * c^4 * \exp(\alpha * L) * \sin(\beta * L) * W - \\ & 8 * c^2 * \alpha^3 * c^3 * \dots \end{aligned}$$

Appendix G Continued

$$\begin{aligned}
 & \beta^4 \exp(\alpha L) \cos(\beta L + 2\alpha W) - \\
 & 64\alpha^3 L \alpha^3 c_1 \dots \\
 & c_2 W \beta^3 + 8\alpha^2 \alpha \beta^4 \nu c_2 c_4 \exp(\alpha L) \sin(\beta L + 2\alpha W) + 32\alpha^3 \alpha \beta^4 \nu c_2 c_3 \exp(\alpha L) \dots \\
 & \sin(\beta L) W + 2\beta^4 c_4^2 \alpha^2 \alpha \cos(2\beta L - 2\alpha W) + \dots \\
 & 2\alpha^2 \alpha \beta^4 c_3^2 \cos(2\beta L + 2\alpha W) - 4\alpha L \dots \\
 & \alpha^4 \sin(2\alpha W) c_4^2 \beta^3 - 16\alpha L \alpha^4 \dots \\
 & \sin(2\alpha W) c_2 c_1 \beta^3 + 8\alpha L \alpha^2 \sin(2\alpha W) \dots \\
 & \beta^5 c_4^2 + 16c_1^2 \alpha^4 \alpha^3 W \beta - 16\alpha^2 \alpha^3 \dots \\
 & \beta^2 c_4^2 c_2 \exp(\alpha L) \sin(\beta L - 2\alpha W) - \\
 & 4c_4^2 \alpha^5 \dots \\
 & \sin(2\beta L) \alpha^3 W - 32\alpha^3 \alpha^2 \beta^3 \nu c_2 c_3 \dots \\
 & \exp(\alpha L) \cos(\beta L) W - 8\alpha^4 c_1 c_4 \exp(- \\
 & \alpha L) \alpha^2 \dots \\
 & \beta \cos(\beta L + 2\alpha W) + 8c_1 \alpha^3 c_4 \beta^4 \exp(- \\
 & \alpha L) \dots \\
 & \sin(\beta L - 2\alpha W) - 16\alpha L \alpha^2 \sin(2\alpha W) c_3^2 \dots \\
 & \beta^5 \nu - 16\alpha L \alpha^2 \sin(2\alpha W) c_4^2 \beta^5 \nu + 8 \dots \\
 & \alpha^3 L \alpha^5 W c_3^2 \beta - \\
 & 8\alpha^4 c_2 c_3 \exp(\alpha L) \alpha \dots \\
 & \beta^2 \cos(\beta L - 2\alpha W) - 32\alpha^3 \alpha \beta^4 \nu c_2 c_4 \dots \\
 & \exp(\alpha L) \cos(\beta L) W - 32\alpha^3 \alpha^4 \beta \nu c_4 c_2 \dots \\
 & \exp(\alpha L) \sin(\beta L) W + 16\alpha^2 \alpha^2 \beta^3 c_2 c_3 \dots \\
 & \exp(\alpha L) \sin(\beta L + 2\alpha W) - \\
 & 8\alpha^3 \alpha \beta^4 c_3^2 \dots \\
 & \sin(2\beta L) W + 16\alpha^2 \alpha^2 \beta^3 c_1 c_3 \exp(-\alpha L) \dots \\
 & \sin(\beta L - 2\alpha W) - \\
 & 8\alpha^4 c_2 c_3 \exp(\alpha L) \alpha^2 \beta \dots \\
 & \sin(\beta L + 2\alpha W) - 16\alpha^2 \alpha^3 \beta^2 c_1 c_4 \dots \\
 & \exp(-\alpha L) \sin(\beta L - 2\alpha W) + 8\alpha^4 c_1 c_3 \exp(- \\
 & \alpha L) \dots \\
 & \alpha^2 \beta \sin(\beta L + 2\alpha W) + 24\alpha^2 \alpha^3 \beta^2 \nu c_1 \dots \\
 & \cdot \\
 & c_4 \exp(-\alpha L) \sin(\beta L - 2\alpha W) - \\
 & 4\alpha^4 c_3 c_4 \alpha^3 \dots \\
 & \sin(2\alpha W) + 2c_3 \beta^4 c_4 \alpha^3 \sin(2\beta L + 2\alpha W) - \\
 & 4 \dots \\
 & \alpha^4 c_1^2 \sin(2\alpha W) \beta \alpha^2 - 16\alpha^3 \alpha^3 \dots \\
 & \beta^2 \nu c_4^2 \sin(2\beta L) W + 16\alpha^3 \alpha \beta^4 c_3 c_4 \dots \\
 & \cos(2\beta L) W + 32\alpha^3 \alpha^3 \beta^2 \nu c_3 c_1 \exp(- \\
 & \alpha L) \dots \\
 & \sin(\beta L) W + 16\alpha^3 \alpha \beta^4 \nu c_3^2 \sin(2\beta L) W + 16 \dots \\
 & \alpha^2 \alpha^2 \beta^3 c_2 c_4 \exp(\alpha L) \cos(\beta L - \\
 & 2\alpha W) + 32 \dots \\
 & \alpha^3 \alpha^2 \beta^3 \nu c_1 c_4 \exp(-\alpha L) \sin(\beta L) W - 32 \dots \\
 & \alpha^5 W c_2 c_3 \alpha^2 \beta - 16\alpha^2 \alpha^3 c_2^2 W \beta^3 - \\
 & c_4^2 \dots \\
 & \alpha^4 \alpha^3 \cos(2\beta L + 2\alpha W) - c_3^2 \alpha^4 \alpha^3 \dots \\
 & \cos(2\beta L - 2\alpha W) - \beta^6 c_3^2 \alpha \cos(2\beta L - \\
 & 2\alpha W) - \dots
 \end{aligned}$$

Appendix G Continued

$$\begin{aligned}
 & \beta^6 c^4 \alpha^2 \cos(2\beta L + 2\alpha W) + c^4 \alpha^2 \alpha^4 \alpha^3 \dots \\
 & \quad \cos(2\beta L - \\
 & 2\alpha W) + \beta^6 c^3 \alpha^2 \cos(2\beta L + 2\alpha W) + \dots \\
 & \quad \beta^4 c^4 \alpha^2 \alpha^3 \cos(2\beta L - 2\alpha W) + 8\alpha^4 c^1 c^3 \dots \\
 & \quad \exp(-\alpha L) \alpha \beta^2 \cos(\beta L + 2\alpha W) - \\
 & 8\alpha^3 \alpha^3 \dots \\
 & \quad \beta^2 c^3 \alpha^2 \sin(2\beta L) W + 32\alpha^3 \alpha^4 \beta \nu c^1 \alpha^2 \dots \\
 & \quad \exp(-\alpha L)^2 W - 4\alpha^4 c^2 \alpha^2 \exp(\alpha L)^2 \beta^3 \dots \\
 & \quad \sin(2\alpha W) + 8\alpha^4 c^1 c^4 \exp(-\alpha L) \alpha^2 \beta^2 \dots \\
 & \quad \cos(\beta L - 2\alpha W) - 32\alpha^3 \alpha^3 \beta^2 \nu c^4 c^2 \dots \\
 & \quad \exp(\alpha L) \cos(\beta L) W + 8\alpha^3 L W c^3 \alpha^2 \beta^5 \alpha^4 \dots \\
 & \quad \alpha^4 c^3 c^4 \alpha \beta^2 \sin(2\alpha W) - \\
 & 2\alpha^4 c^3 c^4 \alpha \dots \\
 & \quad \beta^2 \sin(2\beta L - 2\alpha W) + 2\alpha^4 c^3 c^4 \alpha \beta^2 \dots \\
 & \quad \sin(2\beta L + 2\alpha W) - \\
 & 4\alpha^3 L \alpha^4 \sin(2\alpha W) c^3 \alpha^2 \dots \\
 & \quad \beta^4 L \sin(2\alpha W) \beta^5 c^4 \alpha^2 \alpha^3 - \\
 & 4L \sin(2\alpha W) \dots \\
 & \quad \beta^5 c^3 \alpha^2 \alpha^3 + 8\alpha^4 c^2 c^3 \exp(\alpha L) \alpha^2 \beta^2 \dots \\
 & \quad \sin(\beta L - \\
 & 2\alpha W) + 2\alpha^2 \alpha^3 \beta^2 c^3 \alpha^2 \cos(2\beta L + 2\alpha W) \dots \\
 & \alpha W) + 8\beta^4 c^4 \alpha^2 \alpha^3 \sin(2\beta L) \alpha W + 8\alpha^4 \alpha^2 \dots \\
 & \quad \alpha^4 \beta \nu c^4 c^2 \exp(\alpha L) \cos(\beta L - 2\alpha W) + 8c^1 \dots \\
 & \quad \alpha^3 c^3 \beta^4 \exp(-\alpha L) \cos(\beta L - \\
 & 2\alpha W) + 2c^3 \beta^6 \dots \\
 & \quad c^4 \alpha \sin(2\beta L + 2\alpha W) + 4c^1 \alpha^4 \exp(- \\
 & \alpha L)^2 \dots \\
 & \quad \beta^2 \sin(2\alpha W) - 24\alpha^2 \alpha^2 \beta^3 \nu c^1 c^4 \dots \\
 & \quad \exp(-\alpha L) \cos(\beta L + 2\alpha W) + 16\alpha^3 c^2 \alpha^2 \alpha^2 \dots \\
 & \quad \exp(\alpha L)^2 \beta^3 W - 16\alpha^2 \alpha^3 \beta^2 c^3 c^1 \dots \\
 & \quad \exp(-\alpha L) \cos(\beta L - \\
 & 2\alpha W) + 32\alpha^3 \alpha^2 \beta^3 \nu \dots \\
 & \quad c^1 \alpha^2 \exp(-\alpha L)^2 W - 8\alpha^2 \alpha^4 \beta c^1 \alpha^2 \exp(- \\
 & \alpha L)^2 \dots \\
 & \quad \sin(2\alpha W) - 64\alpha^3 \alpha^3 \beta^2 c^3 c^1 \exp(-\alpha L) \dots \\
 & \quad \sin(\beta L) W - 64\alpha^3 \alpha^3 \beta^2 c^3 c^2 \exp(\alpha L) \dots \\
 & \quad \sin(\beta L) W - 24\alpha^2 \alpha^2 \beta^3 \nu c^2 c^3 \exp(\alpha L) \dots \\
 & \sin(\beta L + 2\alpha W) + 8\beta^2 c^4 \alpha^2 \alpha^3 \sin(2\beta L) \alpha^3 \dots \\
 & \quad W - \beta^4 c^4 \alpha^2 \alpha^3 \cos(2\beta L + 2\alpha W) - 16\alpha^4 \dots \\
 & \quad \sin(2\alpha W) c^1 \alpha^2 c^3 \beta - \\
 & 4L \sin(2\alpha W) \beta^7 c^4 \alpha^2 \dots \\
 & \quad \alpha - 8c^1 \alpha^4 c^3 \beta^3 \exp(-\alpha L) \sin(\beta L + 2\alpha W) - \\
 & \dots \\
 & \quad 16\alpha^3 L \alpha^4 \sin(2\alpha W) c^2 c^1 \beta - 4\alpha^3 L \dots \\
 & \quad \alpha^4 \sin(2\alpha W) c^4 \alpha^2 \beta + 8c^2 \alpha^4 c^3 \beta^3 \dots \\
 & \quad \exp(\alpha L) \sin(\beta L + 2\alpha W) - 32\alpha^3 L \alpha^4 \dots \\
 & \sin(2\alpha W) c^2 c^1 \beta^3 + 8\alpha^3 L \alpha^2 \sin(2\alpha W) \dots \\
 & \quad \beta^3 c^4 \alpha^2 - 16\alpha^3 L \alpha^2 \sin(2\alpha W) c^3 \alpha^2 \beta^3 \dots \\
 & \quad \nu - 16\alpha^3 L \alpha^2 \sin(2\alpha W) c^4 \alpha^2 \beta^3 \nu + 32c^1 \dots \\
 & \quad \alpha^4 c^4 \beta^3 \exp(-\alpha L) \sin(\beta L) W \alpha - \\
 & 8c^3 \beta^6 c^4 \dots \\
 & \quad \cos(2\beta L) \alpha W \alpha - 4c^2 \alpha^6 \exp(\alpha L)^2 \beta^2 \dots
 \end{aligned}$$

Appendix G Continued

```

sin(2*alphan*W)-16*alpha^4*alphan^3*c2^2*W*beta-16*alphan^2*...
sin(2*alphan*W)*beta^4*c1*nu*c4*alpha+32*W*c2*alpha^4*c3*beta^3*...
alphan-32*c1*alpha^3*c3*beta^4*exp(-
alpha*L)*sin(beta*L)*W*alphan+8*...
alpha^3*L*alphan^2*sin(2*alphan*W)*beta^3*c3^2-
4*c3*beta^6*c4*alpha*...
sin(2*alphan*W)-4*beta^6*c4^2*sin(2*beta*L)*alpha*W*alphan-
32*c2*...

alpha^4*c4*beta^3*exp(alpha*L)*sin(beta*L)*W*alphan+8*c2^2*alpha^4*...
exp(alpha*L)^2*beta^3*W*alphan+32*c1*alpha^4*c3*beta^3*exp(-
alpha*L)*...

cos(beta*L)*W*alphan+4*beta^4*c3^2*sin(2*beta*L)*alpha^3*W*alphan+8*...
c2*alpha^4*c4*beta^3*exp(alpha*L)*cos(beta*L-2*alphan*W)-32*c2*...

alpha^4*c3*beta^3*exp(alpha*L)*cos(beta*L)*W*alphan+4*beta^6*c3^2*...
sin(2*beta*L)*alpha*W*alphan-8*c1^2*alpha^4*exp(-
alpha*L)^2*beta^3*...
W*alphan-
8*c3*beta^4*c4*cos(2*beta*L)*alpha^3*W*alphan+8*c1*alpha^4*...
c3*beta^3*exp(-alpha*L)*sin(beta*L-2*alphan*W)-8*c1*alpha^3*c4*...
beta^4*exp(-alpha*L)*sin(beta*L+2*alphan*W))/(alpha^2+beta^2)*...
/alpha/beta/alphan;

```

```

function f=buckling(m,Nx,params)
l=params.L; %Length of Gel
w=params.W; %Width of gel
alphan=m*pi/w;
nu=params.nu; %Poisson Ratio
Def=params.D; %Flexural Rigidity
alpha=sqrt(m^2*pi^2/(w^2)+sqrt(Nx/Def*m^2*pi^2/(w^2)));
beta=sqrt(-m^2*pi^2/(w^2)+sqrt(Nx/Def*m^2*pi^2/(w^2)));

A=1;
B=1;
C=1;
D=0;
E=-alpha;
F=alpha;
G=0;
H=beta;
I=(alpha^2*exp(-alpha*l)-alphan^2*nu*exp(-alpha*l));
J=(alpha^2*exp(alpha*l)-alphan^2*nu*exp(alpha*l));
K=(-beta^2*cos(beta*l)-alphan^2*nu*cos(beta*l));
L=(-beta^2*sin(beta*l)-alphan^2*nu*sin(beta*l));
M=(-alpha^3*exp(-alpha*l)+alphan^2*(2-nu)*alpha*exp(-alpha*l));
N=(alpha^3*exp(alpha*l)-alphan^2*(2-nu)*alpha*exp(alpha*l));
O=(beta^3*sin(beta*l)+alphan^2*(2-nu)*beta*sin(beta*l));
P=(-beta^3*cos(beta*l)-alphan^2*(2-nu)*beta*cos(beta*l));
%Determinant of 4x4 Matrix
f=A*(F*(K*P-O*L)-G*(J*P-L*N)+H*(J*O-K*N))-...
B*(E*(K*P-O*L)-G*(I*P-L*M)+H*(I*O-K*M))+...

```

Appendix G Continued

```
C*(E*(J*P-L*N)-F*(I*P-L*M)+H*(I*N-J*M))-...  
D*(E*(J*O-K*N)-F*(I*O-K*M)+G*(I*N-J*M));
```

```
function C=buckling2(m,Nx,params)  
L=params.L; %Length of Gel  
W=params.W; %Width of gel  
alphan=m*pi/W;  
nu=params.nu; %Poisson Ratio  
Def=params.D; %Flexural Rigidity  
alpha=sqrt(m^2*pi^2/(W^2)+sqrt(Nx/Def*m^2*pi^2/(W^2)));  
beta=sqrt(-m^2*pi^2/(W^2)+sqrt(Nx/Def*m^2*pi^2/(W^2)));  
  
A=[1 1 1 0; -alpha alpha 0 beta;...  
 (alpha^2*exp(-alpha*L)-alphan^2*nu*exp(-alpha*L)) ...  
 (alpha^2*exp(alpha*L)-alphan^2*nu*exp(alpha*L)) ...  
 (-beta^2*cos(beta*L)-alphan^2*nu*cos(beta*L)) ...  
 (-beta^2*sin(beta*L)-alphan^2*nu*sin(beta*L));...  
 (-alpha^3*exp(-alpha*L)+alphan^2*(2-nu)*alpha*exp(-alpha*L)) ...  
 (alpha^3*exp(alpha*L)-alphan^2*(2-nu)*alpha*exp(alpha*L)) ...  
 (beta^3*sin(beta*L)+alphan^2*(2-nu)*beta*sin(beta*L)) ...  
 (-beta^3*cos(beta*L)-alphan^2*(2-nu)*beta*cos(beta*L))];  
c=[0;0;0;0];  
Amat=[A c];  
%Making C4=1  
Aref=rref(Amat);  
C=[-Aref(1,4);-Aref(2,4);-Aref(3,4);1];
```